Bisimulation and Propositional Intuitionistic Logic

Anna Patterson
Computer Science Department, Stanford University, Stanford, CA 94305, Email annap@cs.stanford.edu

Abstract

Brouwer suggests that $p \supset q$ can be interpreted as a computation that given a proof of p constructs a proof of q. Dually we can see this as saying that every counter model of q contains a counter model of p. However, what does it mean to contain another Kripke structure? We read this as: every counter model of q contains a substructure that is bisimilar to a counter model of p.

If q and p are interderivable, then the counter models contain each other. Thus we study *bisimulation* as a way of studying interderivability. Using bisimulation, we are able to characterize validity in a Kripke structure.

Theorem 1 Let K be a *finite* Kripke structure for propositional intuitionistic logic, then two worlds in K are bisimilar if and only if they satisfy the same set of formulas.

This theorem lifts to structures in the following manner.

Theorem 2 Two *finite* Kripke structures K and K' are bisimilar if and only if they have the same set of valid formulas.

1 Background

The standard model theory for intuitionistic logic [5] was introduced by Kripke [11]. This model theory was originally introduced as a characterization of modal logics. A natural question that arises for any model theory is characterizing when two models satisfy the same set of formulas. A partial answer to this for modal logics was given by van Benthem, who showed that for well behaved modal structures the notion of bisimulation was appropriate.

We now apply the same techniques to intuitionistic logic. Although intuitionistic logic is not a normal modal logic we can use the relationship between S4 and topology to provide a bridge.

The algebra of S4 [12] is a topological boolean algebra [14] where the modality $\Box$ can be taken to be an interior operation and the modality $\Diamond$ is taken to be a closure operation [13, 18]. It is well known [4] that the open subsets of a topological boolean algebra form a relatively pseudo-complemented lattice. However, a relatively pseudo-complemented lattice with a bottom is a pseudo-complemented lattice, that is a Heyting algebra and hence an algebra of intuitionistic logic.

Following Gödel we can see intuitionistic logic as a fragment of S4.

We prove that the finite Kripke structures for intuitionistic logic are bisimilar exactly when they entail the same set of intuitionistic formulas. Finally, this means that two transitive monotonic modal models are bisimilar if and only if their associated finitely axiomatizable sets of valid intuitionistic formulas are equal. This can be interpreted as saying that all functions between open sets, the $\Box$-formulas, are definable in intuitionistic logic [15]. Since functions between open sets are continuous, this result is perhaps better known in computer science as: continuous functions are computable [22].

2 Intuitionistic Logic

We present a model theory for intuitionistic logic due to Kripke. For a nice overview of this model theory see Fitting [7].

Definition 1 *A Kripke structure for propositional intuitionistic logic over a set P of propositional atoms is a triple $\langle W, R, V\rangle$ where W is a non-empty set (the set of* worlds*), R is a binary reflexive and transitive relation on W and V is a function assigning to every world a set of propositional atoms where the assignment V satisfies, for every world w, $wRw' \to V(w) \subseteq V(w')$.*

Thus, V is monotonic on propositional atoms.

Definition 2 *The satisfiability relation $\models_K$ on a Kripke structure $K = \langle W, R, V\rangle$ over P is a relation between worlds and propositional formulas. We define this relation inductively over the structure of a formula θ. Thus, for any $w \in W$,*

$$
\begin{array}{lllr}
\theta = p & w \models_K p & \equiv & V(w,p) = 1 \\
\theta = \varphi \wedge \psi & w \models_K \varphi \wedge \psi & \equiv & w \models_K \varphi \textit{ and } w \models_K \psi \\
\theta = \varphi \vee \psi & w \models_K \varphi \vee \psi & \equiv & w \models_K \varphi \textit{ or } w \models_K \psi \\
\theta = \varphi \supset \psi & w \models_K \varphi \supset \psi & \equiv & \forall w' \in W.\ \ w\,R\,w' \ \wedge\ w' \models_K \varphi \to w' \models_K \psi \\
\theta \sim \varphi & w \models_K \ \sim \varphi & \equiv & \forall w' \in W.\ \ w\,R\,w' \ \to\ w' \not\models_K \varphi
\end{array}
$$

We say that a formula φ is satisfied at a world w in a structure K if $w \models_K \varphi$.

Definition 3 *Let $K = \langle W, R, V\rangle$ be a Kripke structure, then a formula φ is* satisfied *in K if and only if $\exists w \in W.\ \ w \models_K \varphi$. A formula φ is* valid *in K if and only if $\forall w \in W.\ \ w \models_K \varphi$. A formula φ is* valid *if and only if φ is valid in all Kripke structures.*

When we say a set of formulas Θ is *conjunctively satisfied* in K we mean that there is a world in K which simultaneously satisfies all of Θ. Similarly we say a set of formulas Θ is *disjunctively satisfied* when for every world in K there is a formula in Θ which is satisfied. Because intuitionistic logic does not contain classical negation these notions are independent.

3 Bisimulation within a structure

Bisimulation was introduced by Milner [16] as a way of defining an equivalence notion for a process algebra. There bisimulation characterized equivalences between labeled transition systems. Earlier, van Benthem showed the close connection between bisimulation (or more precisely *p-morphisms*) and definability in modal logics [3]. Using the classical transformation of modal formulas to first order formulas, he showed that a first order formula is preserved under bisimulation exactly when it is equivalent to the translation of a modal formula. Bisimulation has been used in a variety of other ways; Aczel [1] for instance showed that bisimulation provided a natural identity condition for non-well-founded sets.

Bisimulation is usually defined over labeled transition systems. In Kripke structures for intuitionistic logic, we do not really have labels on transitions. Therefore, we consider labeled transition systems with just one label; in addition, the states of these labeled transition systems are decorated with the set of propositional atoms "true" at that Kripke world.

The following is the definition of bisimulation equivalence over a such a structure.

Definition 4 *Let $K = \langle W, R, V\rangle$ be a Kripke structure, then we say that two worlds $w, w' \in W$ are* bisimilar*, denoted $w \leftrightarroweq w'$, in K if there exists a symmetric relation $B \subseteq W \times W$ (a* bisimulation*) such that*

$$\begin{array}{lll} wBw' & & \wedge \\ xBy & \Rightarrow V(x) = V(y) & \wedge \\ xBy & \Rightarrow \forall x'.\ (x\,R\,x' \rightarrow \exists y'.\ y\,R\,y' \ \wedge x'\,B\,y') & \end{array}$$

The set of equivalence classes under $\leftrightarroweq$ is denoted, $W/\leftrightarroweq$ and the equivalence class of w is denoted $[w]$.

We can make a smaller structure using this equivalence relation on worlds.

Lemma 5 *Let $K = \langle W, R, V\rangle$ be a Kripke structure, then $K/\leftrightarroweq\ = \langle W/\leftrightarroweq, R/\leftrightarroweq, V/\leftrightarroweq\rangle$ is a Kripke structure where*

- $[w]R/\leftrightarroweq [w'] \equiv \exists w_1 \in [w], \exists w'_1 \in [w']$ *such that* $w_1\,R\,w'_1$, *and*
- $V/\leftrightarroweq([w]) = V(w)$

Proof: To show that $K/\leftrightarroweq$ is a Kripke structure, we need only show that $R/\leftrightarroweq$ is a reflexive transitive relation and that $V/\leftrightarroweq$ is monotonic with respect to propositional atoms. These follow by easy application of the definition of K and $\leftrightarroweq$. ∎

The following theorem states that bisimulation is truth preserving and does not strengthen the structure to prove any extra formulas. The forward direction of this theorem appears as a lemma in Rodenberg's thesis [21]. The backward direction is an easy exercise in structural induction.

Theorem 6 *Let $K = \langle W, R, V \rangle$ be a Kripke structure, then $\forall w \in W$,*

$$w \models_K \alpha \text{ if and only if } [w] \models_{K/\underline{\leftrightarrow}} \alpha.$$

Definition 7 *Let $K = \langle W, R, V \rangle$ be a Kripke structure, then w_1 and w_2 are* elementarily equivalent*, denoted $w_1 \approx w_2$, if for all formulas φ, $w_1 \models_K \varphi$ if and only if $w_2 \models_K \varphi$.*

We say that a Kripke structure is *finite* if the cardinality of W is finite. Studying finite Kripke structures is not particularly restrictive because it is well known that an intuitionistic formula is valid if and only if it is valid in all finite Kripke structures. Henceforth a Kripke structure refers to a Kripke structure for propositional intuitionistic logic over a *finite* set of propositional atoms P. Note the distinction between using a finite set of propositional atoms, and using a finite Kripke structure.

Next we show that if two worlds in a finite Kripke structure are elementarily equivalent, then they are bisimilar. To the best of our knowledge this result is new here.

Theorem 8 *Let $K = \langle W, R, V \rangle$ be a* finite *Kripke structure, then $w_1 \approx w_2$ implies that $w_1 \underline{\leftrightarrow} w_2$.*

Proof: We want to show that $\approx$ is a bisimulation. Clearly $\approx$ is symmetric. Thus we want to show that if $w_1 \approx w_2$ then $V(w_1) = V(w_2)$ and that for all $w_1' \in W$ such that $w_1 \, R \, w_1'$ there is a $w_2' \in W$ such that $w_2 \, R \, w_2'$ and $w_1' \approx w_2'$.

Given two worlds such that $w_1 \approx w_2$, by definition $V(w_1) = V(w_2)$. Let $w_1 R w_1'$ and let Θ be the set of formulas satisfied at w_1'.

We now show that (1) every formula in Θ must be satisfied at some world accessible from w_2, (2) that there is at least one world accessible from w_2 which satisfies every formula in Θ, (3) that there is a world accessible from w_2 which satisfies no formulas other than Θ.

1. Let $\varphi \in \Theta$. If there were no world accessible from w_2 which satisfied φ, then the formula $\sim \varphi$ would be satisfied at w_2. This cannot be the case because $w_1' \models_K \varphi$ thus $w_1 \not\models_K \sim \varphi$, and w_1 and w_2 are elementarily equivalent. This shows that every formula in Θ is satisfied at some world w_2' where $w_2 R w_2'$.

2. Now we need to ensure that all $\varphi \in \Theta$ are satisfied at some w_2' accessible from w_2. We do so by defining a formula $\gamma \in \Theta$ which *in K* implies every other formula in Θ.

   Let $W_<$ be the set of worlds in K that do not satisfy all of Θ. For each world w in $W_<$, pick some formula ϕ_w in Θ such that $w \not\models_K \phi_w$. As $W_<$ is finite, let γ be the finite conjunction $\bigwedge_{w \in W_<} \phi_w$. In K, by construction, in every world where γ is satisfied, all of Θ is satisfied.

We know that $w'_1 \models_K \gamma$ thus $w_1 \not\models_K \sim \gamma$. Since $w_1 \approx w_2$, it follows that $w_2 \not\models_K \sim \gamma$. This implies that there is a world w'_2 accessible from w_2 that satisfies all of Θ.

3. Let $W_>$ be the set of worlds in K that satisfy a formula not in Θ. For each world w in $W_>$, pick some formula ψ_w which is not in Θ such that $w \models_K \psi_w$. As $W_>$ is finite, let δ be the finite disjunction $\bigvee_{w \in W_>} \psi_w$. In K, by construction, in every world where δ is refuted, only formulas in Θ are satisfied.

   We know that $w'_1 \models_K \gamma$ but $w'_1 \not\models_K \delta$. So $w_1 \not\models_K \gamma \supset \delta$. Since $w_1 \approx w_2$, $w_2 \not\models_K \gamma \supset \delta$. Therefore there is a world accessible from w_2 that satisfies γ and refutes δ. This means that there is a world accessible from w_2 which satisfies exactly Θ.

∎

Corollary 9 *Let $K = \langle W, R, V \rangle$ be a finite Kripke structure, then for all $w, w' \in W$.*

$$w \,\underline{\leftrightarrow}\, w' \text{ if and only if } w \approx w'$$

Proof: This follows from Theorem 6 and 8 ∎

We now extend this result to a larger class of structures. We first discuss this class, and then prove a theorem analogous to Corollary 9.

Definition 10 *Let $K = \langle W, R, V \rangle$ be a Kripke structure. K has* finite principal filters *when for every world w in W the principal filter, that is the set $\uparrow(w) = \{w' \mid w\, R\, w'\}$, is finite.*

Making the set of worlds accessible from any world finite seems like a quite restrictive condition. However, this condition does not imply that the entire structure is finite for there could exist a infinite set of incomparable worlds. Finite principal filter structures have the advantage over finite structures in that they can describe a canonical model for any intuitionistic theory over a finite number of propositional atoms.

Definition 11 *A Kripke structure K is a* canonical *model for propositional intuitionistic logic if every valid propositional intuitionistic formula is valid in K and K refutes every underivable formula.*

We call such structures *canonical Kripke models.*

It follows from the work of Riger and Nishimura [19, 17] that there are no *finite* canonical Kripke models for propositional intuitionistic logic even if the cardinality of the set of propositional atoms is only 1. However, there is a canonical Kripke model with the finite principal filter property. For each *finite* set of propositional atoms [2] constructed such a model which moreover has the nice property that no two distinct worlds in this structure are bisimilar. In other words, this structure is the bisimulation class representative.

Definition 12 *Let $K = \langle W, R, V\rangle$ be a Kripke structure, and let w be an element of W, then $K \uparrow w$ is the Kripke structure, $\langle \uparrow(w), R\restriction_{\uparrow(w)}, V\restriction_{\uparrow(w)}\rangle$.*

This is simply the restriction of K to the worlds above w. Technically we should prove that $K \uparrow w$ is a Kripke structure, but this is immediate. This definition can be easily generalized to consider a set $X \subseteq W$ of elements to construct a structure $K \uparrow X$ from the filter generated by X which is $\bigcup_{w \in X} \uparrow(w)$.

Theorem 13 *Let $K = \langle W, R, V\rangle$ be a Kripke structure with* finite principal filters*, then $w_1 \approx w_2$ implies that $w_1 \leftrightarroweq w_2$.*

Proof: Consider the filter generated by $\{w_1, w_2\}$. This equals $\uparrow(w_1) \cup \uparrow(w_2)$. Since $\uparrow(w_1)$ is finite and $\uparrow(w_2)$ is finite, also $\uparrow(w_1) \cup \uparrow(w_2)$ is finite. We apply Theorem 8 to conclude that $w_1 \leftrightarroweq w_2$ in $K \uparrow \{w_1, w_2\}$. However, a bisimulation between w_1 and w_2 in $K \uparrow \{w_1, w_2\}$ is also a bisimulation in K. ∎

Corollary 14 *Let $K = \langle W, R, V\rangle$ be a Kripke structure with* finite principal filters*, then for all $w, w' \in W$.*

$$w \leftrightarroweq w' \text{ if and only if } w \approx w'$$

Proof: This follows from Theorem 6 and 13 ∎

Now we attempt to generalize more in the lines of what has been accomplished in the modal setting, see Hollenberg [10] for an excellent overview and de Rijke [20] for a collection of papers pertinent to the following generalizations.

Definition 15 *Let $K = \langle W, R, V\rangle$ be a Kripke structure, then K is* conjunctively saturated *if for every world w and for every set of formulas Θ, whenever $K \uparrow w$ conjunctively satisfies every finite subset of Θ, then $K \uparrow w$ conjunctively satisfies Θ.*

We say that K is disjunctively saturated *whenever $K \uparrow w$ disjunctively satisfies Θ, then $K \uparrow w$ disjunctively satisfies some finite subset of Θ.*

We say that a structure is *saturated* if it is conjunctively saturated and disjunctively saturated. Since intuitionistic logic does not obey the de Morgan laws, we need the structures to be saturated conjunctively and disjunctively.

Theorem 16 *Let $K = \langle W, R, V\rangle$ be a* saturated *Kripke structure, then $w_1 \approx w_2$ implies that $w_1 \leftrightarroweq w_2$.*

Outline of Proof: We use the proof of Theorem 8 in the following manner. (1) is still valid. We use conjunctive saturatedness and (1) to prove (2) and disjunctive saturatedness to prove (3). ∎

Another way of stating this result which may be more obviously related to the proof of Theorem 8 is that for every world w in a saturated structure, w satisfies only a finitely axiomatizable theory. This means that for every world w we can construct a γ formula which *in* $K$is equivalent to the theory of w. Then we can see that Theorem 8 immediately generalizes to a proof that every finite principal filter structure is saturated.

Another result in modal logics is that bisimulation and elementary equivalence correspond for image finite structures. There are two manners in which we could define image finite structures. When we view Kripke structures as annotated labeled transitions systems with only a single label then, because of transitivity, image finite structures are simply the finite principal filter structures. In the preceding text, we have just studied this form of image finite structure. Another quite reasonable reading of an image finite structure requires an alternate transformation from a Kripke structure to a labeled transition system. This alternate transformation labels transitions between Kripke worlds with the set of propositional atoms that have "just become true". That is for each pair of worlds w, w' such that wRw' let us label the transition with the difference among the propositional atoms.

Definition 17 *A Kripke structure,* $K = \langle W, R, V \rangle$ *is* image finite *when for every world* w *and for every propositional atom* p*, the set* $\{w' \mid p \in V(w') - V(w) \wedge w' \text{ is an immediate successor of } w\}$ *is finite.*

Using the previous definition, the following serves as a counterexample to the proposed result that two image finite structures with the same theory are bisimilar. Note the following counterexample has only a finite number of propositional atoms and needs the following theorem due to [2].

Theorem 18 *If the number of propositional atoms is greater than 1, then every canonical Kripke model has an infinite anti-chain.*

Fact 19 Counterexample *Take the canonical finite principal filter Kripke structure on two generators, denoted* K_2*, and add a bottom world* w_* *from which the rest of the structure is accessible and whose valuation is the empty set.*

Now for all infinite subsets of the infinite anti-chain of K_2 *(which exists by Theorem 18) take the infinite meet. At each infinite meet, we add a world to the Kripke structure whose valuation is the empty set. Then add another bottom world* w'_* *from which the entire structure constructed thus far is accessible and whose valuation is the empty set.*

In this structure, w_* and w'_* both have the same theory. The theory *true* to be exact. However w_* and w'_* are not bisimilar. **Note**, the worlds in the infinite anti-chain all have the empty set as the valuation, so no atom becomes true along those transitions.

4 Bisimulation between structures

The definition of bisimulation can be extended slightly to include the equivalence of two worlds from different structures.

Definition 20 *Let $K = \langle W, R, V\rangle$ and $K' = \langle W', R', V'\rangle$ be two Kripke structures over P, then K is bisimilar to K', $K \leftrightarroweq K'$, if there is a symmetric relation $S \subseteq W \times W'$ such that $\forall w \in W.\ \exists w' \in W'$ with $w\ S\ w'$ and*

$$x\ S\ y \Rightarrow V(x) = V'(y) \quad \wedge \quad x\ S\ y \Rightarrow \forall x' \in W.\ x\, R\, x' \rightarrow \exists y' \in W'.\ \ y\, R'\, y' \wedge x'\ S\ y'$$

Definition 21 *Let $K = \langle W, R, V\rangle$ and $K' = \langle W', R', V'\rangle$ be two Kripke structures over P. We say that K is isomorphic to K', denoted $K \cong K'$, if there is a bijection on worlds, call this function $f\colon W \rightarrow W'$, such that*

- *$\forall w, w' \in W.\ \ w\, R\, w'$ if and only if $f(w)\ R'\ f(w')$, and*
- *$\forall w \in W.\ \ V(w) = V'(f(w))$.*

The following is a corollary to Theorem 9 (Theorem 14).

Corollary 22 *Let $K = \langle W, R, V\rangle$ be a finite(finite principal filter) Kripke structure, then $K/\!\leftrightarroweq\ \cong K/\approx$.*

Lemma 23 *Let $K = \langle W, R, V\rangle$ and $K' = \langle W', R', V'\rangle$ be two Kripke structures over P, then K and K' are bisimulation equivalent, $K \leftrightarroweq K'$, if and only if $K/\!\leftrightarroweq$ is isomorphic to $K'/\!\leftrightarroweq$.*

Outline of Proof: From left to right we are given a symmetric relation from which we construct the bijection in the obvious manner and from right to left we are given a bijection from which we construct the symmetric relation. ∎

Theorem 24 *Let $K = \langle W, R, V\rangle$ and $K' = \langle W', R', V'\rangle$ be two finite Kripke structures over P, then K and K' have the same set of valid formulas, $K \approx K'$, if and only if $K/\approx$ is isomorphic to $K'/\approx$.*

Outline of Proof: From left to right we have that the two structures prove the exactly the same set of formulas at every world, thus they have the same set of valid formulas.

From right to left we expand the definition for $K \approx K'$ which gives $\{\varphi \mid \forall w \in K.\ \ w \models_K \varphi\} = \{\psi \mid \forall w' \in K'.\ \ w' \models_{K'} \psi\}$.

However, by monotonicity, the set of valid formulas in a finite structure is just the intersection of the formulas satisfied at the minimal worlds. By Smorynski's transformation, without loss of generality, we may consider adding a point to K to obtain K_* and a point to K' to obtain K'_*. Since $K \approx K'$, we have that $K_* \approx K'_*$. This means that the minimal worlds have the same theory. Thus by Theorem 8 the minimal worlds are bisimilar.

If the minimal worlds are bisimilar, then the structures are bisimilar by Definition 20. Thus by Lemma 23, $K/\leftrightarroweq \;\cong\; K'/\leftrightarroweq$. But for finite structures K, by Corollary 22, $K/\leftrightarroweq \;\cong\; K/\approx$, thus $K/\approx \;\cong\; K'/\approx$. ∎

Corollary 25 *Let $K = \langle W, R, V\rangle$ and $K' = \langle W', R', V'\rangle$ be two finite Kripke structures over P, then the following are all equivalent,*

$$(1)\ K \approx K' \quad (2)\ K \leftrightarroweq K' \quad (3)\ K/\leftrightarroweq \;\cong\; K'/\leftrightarroweq$$

$$(4)\ K/\approx \;\cong\; K'/\approx \quad (5)\ K/\approx \;\cong\; K'/\leftrightarroweq$$

∎

Corollary 26 *Let $K = \langle W, R, V\rangle$ and $K' = \langle W', R', V'\rangle$ be two finite principal filter Kripke structures over finite set of propositional atoms P, then the following are all equivalent,*

$$(1)\ K \approx K' \quad (2)\ K \leftrightarroweq K' \quad (3)\ K/\leftrightarroweq \;\cong\; K'/\leftrightarroweq$$

$$(4)\ K/\approx \;\cong\; K'/\approx \quad (5)\ K/\approx \;\cong\; K'/\leftrightarroweq$$

∎

5 Conclusion

Bisimulation has been examined in conjunction with modal logics starting with van Benthem's correspondence theory [3]. He stated a similar result, that two finite Kripke structures for modal systems have the same set of valid formulas if and only if the two Kripke structures are bisimilar. Although the forward direction of our correspondence (Theorem 6) can be obtained as a corollary of the corresponding results for modal logic, the backwards direction (Theorem 8) strengthens the traditional modal results. Goldblatt has also studied validity preserving constructions for modal logics and a survey is found in [8].

We considered the same problem for intuitionistic logic. This is non-trivial, as the translations from intuitionistic logic into modal logics [6] are not onto. There is further difficulty because it has been shown that bisimulation is characterized by $\diamond$ and negation and conjunction. However there is no definable intuitionistic connective which corresponds to $\diamond$ nor to classical negation. Therefore we would expect that a stronger notion than bisimulation to characterize intuitionistic logic. Two structures could differ on a modal formula but not on any intuitionistic formula.

There is however extra structure in intuitionistic models, namely monotonicity. There are very deep connections between monotonicity, computability and intuitionistic logic [13, 15, 4, 22]. It is this connection that allows the reduct of the models to the monotonic ones to exactly parallel the reduct of the modal logic to intuitionistic logic. Restricting our language to intuitionistic logic and restricting our model to open sets is orthogonal to elementary equivalence and thus bisimulation.

Nonetheless, we have obtained a correspondence between elementary equivalence and bisimulation equivalence for finite structures, for finite principal filter structures and for saturated structures.

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