A Logical Treatment of Constructive Duality

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Abstract

We present an investigation of duality in the traditional logical manner. We extend Nelson's symmetrization of intuitionistic logic, constructible falsity, to a self-dual logic; constructible duality. We develop a self-dual model by considering an interval of worlds in an intuitionistic Kripke model. The duality arises through how we judge truth and falsity. Truth is judged forward in the Kripke model, as in intuitionistic logic, while falsity is judged backwards, that is forward in the dual model.

We define a symmetrization of the Beth-Fitting construction which transforms an interval Kripke model in a self-dual algebra for the logic of constructible duality and back. We then show that every point in the algebra is representable by some formula in the logic.

This algebra arises as an instantiation of a pseudo-Boolean algebra into several categorical constructions. In particular, we show that this algebra is an instantiation of the **Chu** construction applied to a pseudo-Boolean algebra, the second Dialectica construction applied to a pseudo-Boolean algebra, and as posetal case of a suggestion by Pitts for the study of recursion and corecursion. Thus the algebra provides a common base for these constructions, and suggests itself as a necessary part of any logical treatment of constructive duality.

1 Statement of the Issues

Duality, symmetry, and categorical opposites play an increasing role in our understanding of fundamental structural properties of computation. Recent investigations have focussed on powerful categorical tools, that provide great expressive power, such as the **Chu**, the dialectica and special final coalgebra constructions, in contrast to some traditional logical methods. This paper attempts to apply old fashioned logical techniques, such as Kripke models, Heyting algebras, Beth and Beth-Fitting constructions to investigate these issues of duality.

Our study begins with Nelson's logic of constructible falsity N which symmetrizes intuitionistic logic by adding an involutive constructive negation $\neg$. The constructive negation of a formula φ, $\neg\varphi$, should be seen as an effective construction of a counterexample to φ. We modify constructible falsity by dropping $\varphi \wedge \neg\varphi \supset \psi$ and by adding two new independent constants $\perp$ and $\top$. These slight modifications to N make this logic both constructive and one in which every operator has a dual; hence the name constructible duality.

We then develop a new self-dual semantics for this logic. Models for logics can be more than equivalence classes of formula under the derivability relation. A classic example which illustrates this difference is Kripke models for modal logic. Kripke models brought with them a philosophical viewpoint, that of "possible worlds". Judging truth from a world in a Kripke model, represents the view from that possible world. From this vantage point other viewpoints are possible when other worlds are accessible. In models of intuitionistic logic, the accessibility relation flows in the direction of time. As time progresses, more base facts may be learned. Thus a model theory may bring more than just a completeness result, it may shed light on the underlying phenomena.

To this end, we break from this philosophical viewpoint of possible worlds and we allow instead for some indeterminacy about which world is the present world. We consider that there are some facts about the world which are certainly true and some facts which are definitely false, leaving open the possibility of conflict and the possibility of indecision. This leaves us uncertain about which possible world is our exact location. This region of uncertainty, we call an interval.

Once we develop a new model theory, we are faced with the question of assessing its suitability. The naturality of a model can often be seen through the translations between the models and the algebra. The translations relating these approaches can verify how close the model is to the structural features of the logic. This is one of the most important studies in model theory. For classical propositional logic, we have Stone duality, in category theory the Yoneda embedding, and in intuitionistic logic the Beth-Fitting construction. We consider symmetric versions of this type of construction.

Duality has recently come to the forefront of logical investigations in computer science. Linear logic, corecursion, the μ-calculus, and self-dual categories have attracted much attention. Here we consider many of these same structures, but we start from a logical perspective, where duality appears in the form of an operator, constructive negation and two constants, undefined and overdefined.

2 Summary of main results

We propose an extension from N to constructible duality, and propose a self-dual Kripke structure for which we show that constructible duality is sound and complete.

We note that the Beth-Fitting construction is similar to an instance of the Yoneda embedding and define a symmetric version of Beth-Fitting construction which inter-translates the interval Kripke models and a self-dual algebra $H \times H^{op}$, where H is a pseudo-Boolean algebra.

We show that the free algebra of constructible duality over n generators is the free algebra for intuitionistic logic over n generators times its opposite. Every point in this algebra is represented by a formula in the logic.

If we symmetrize the Beth construction and apply it to a interval Beth model, the symmetric Beth construction becomes an instantiation of a symmetric Yoneda embedding.

Finally we show that this algebra is connected via isomorphisms to an instance of the **Chu** construction and a special subcategory of the G dialectica category. This provides evidence that these models for constructible duality are models of linear logic with mix. We verify this theorem independently by using a theorem by Seely(1989).

3 Significance and Relevance

Intuitionistic logic has as its basis a constructive proof, or a constructive demonstration. It is this feature which makes it important in computer science. The constructive nature of proofs in this logic mean that proof normalization corresponds to computation in the typed-lambda calculus via the Curry-Howard isomorphism. Realizability interpretations make it possible to extract programs from intuitionistic proofs of existence (see (Mints 1992) for a proof that most realizability interpretations are equivalent).

The notion that a construction fails is the only negation offered in intuitionistic logic. Constructive mathematics preserves the symmetry of Boolean logic's two truth values when it allows a proposition to be undecided: "could be either true or false". Nelson's system N(Nelson 1949) of constructible falsity restores symmetry to intuitionistic logic with an involutory *constructive* negation $\neg\varphi$ asserting the effective construction of a counterexample to φ. (cf. Zaslavskiĭ(1973) for another symmetrization.)

Duality and symmetry are prevalent in the study of linear logic and concurrency(de Paiva 1989, de Paiva and Hyland 1993, Gupta 1994, Pratt 1995). The duality in linear logic arises from the presence of a constructive involutory negation and the symmetric nature of proofs (Girard 1987). The μ calculus(Kozen 1983) is another natural dual logic where its negation acts as a dualizing operator, turning least fixed points to greatest fixed points. The μ-calculus is not constructive but has many of the features present in constructive duality.

Least and greatest fixed point constructions are central to computer science. The partial computable functions are naturally those defined via least fixed points on recursive operators. (Patterson and Costello 1996) have recently proposed using combinations of computable greatest and least fixed points as a programming constructs. Pitts (1994), Milner (1991), Fiore (1996) and others have argued for the importance of corecursion, or greatest fixed points. Pitts, in (1994) stresses the need for mixing recursion and corecursion, and suggests that $C^{op} \times C$ is an important candidate for these investigations. Our algebra is the posetal case of his suggestion.

We show that the underlying structure, present in these formalisms has $H \times H^{op}$ as a simple case; hence constructible duality as a part of its reasoning machinery.

4 Intuitionistic Logic

Gentzen System Sequents $\Gamma \Rightarrow \Delta$ should be read $\wedge\Gamma \supset \vee\Delta$, where $\Gamma \cup \Delta$ is a set of formulas.

Axiom

$$(1)\Gamma, \varphi \Rightarrow \Delta, \varphi$$

Rules of Inference

$$\Rightarrow \wedge \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \qquad \wedge \Rightarrow \quad \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$$

$$\vee \Rightarrow \quad \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \qquad \Rightarrow \vee \quad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$\supset\Rightarrow \quad \frac{\varphi \supset \psi, \Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \supset \psi, \Gamma \Rightarrow \Delta} \qquad \Rightarrow\supset \quad \frac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$$

$$\boxed{\sim\Rightarrow \quad \frac{\sim \varphi, \Gamma \Rightarrow \Delta, \varphi}{\sim \varphi, \Gamma \Rightarrow \Delta} \qquad \Rightarrow\sim \quad \frac{\varphi, \Gamma \Rightarrow}{\Gamma \Rightarrow \Delta, \sim \varphi}}$$

The boxed rules do not appear in the positive fragment of intuitionistic logic, denoted I_p.

5 Adding a Constructive Negation

Adding the following rules to those for I_p yields the system of N^-.

$$\neg\neg \Rightarrow \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\neg\neg\varphi, \Gamma \Rightarrow \Delta} \qquad \Rightarrow \neg\neg \quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi}$$

$$\neg\wedge \Rightarrow \quad \frac{\neg\varphi, \Gamma \Rightarrow \Delta \quad \neg\psi, \Gamma \Rightarrow \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta} \qquad \Rightarrow \neg\wedge \quad \frac{\Gamma \Rightarrow \Delta, \neg\varphi, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \wedge \psi)}$$

$$\neg \supset\Rightarrow \quad \frac{\varphi, \neg\psi, \Gamma \Rightarrow \Delta}{\neg(\varphi \supset \psi), \Gamma \Rightarrow \Delta} \qquad \Rightarrow \neg \supset \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \supset \psi)}$$

Note $\varphi \vee \psi$ is definable as $\neg(\neg\varphi \wedge \neg\psi)$ and with a constant $\mathbf{0}$, then $\sim \varphi = \varphi \supset \mathbf{0}$.

To get *constructible duality* add to N^-, the following rules for constants (Hilbert style) as follows: $\varphi \supset \mathbf{1}$ $\quad \mathbf{0} \supset \varphi$ $\quad (\neg\mathbf{0}) \equiv \mathbf{1}$ $\quad (\neg\mathbf{1}) \equiv \mathbf{0}$ $\quad \varphi \supset \top$ $\quad \bot \supset \varphi$ $\quad (\neg\top) \equiv \top$ $\quad (\neg\bot) \equiv \bot$ where $\mathbf{0}$ is falsity, $\mathbf{1}$ is truth, $\bot$ is undefined and $\top$ is overdefined and $\varphi \equiv \psi$ means $\varphi \supset \psi$ and $\psi \supset \varphi$.

6 Kripke structures for Intuitionistic Logic

Definition 1 *A Kripke structure for propositional intuitionistic logic over a set P of propositional atoms is a triple, $\langle W, R, V\rangle$, where W is a non-empty set (the set of* worlds*), R is a binary reflexive and transitive relation on W and V is a function assigning to every world a set of propositional atoms satisfying for every world w, $wRw' \rightarrow V(w) \subseteq V(w')$.*

Thus, V is monotonic on propositional atoms.

Definition 2 *The satisfiability relation* $\models_K$ *on a Kripke structure* $K = \langle W, R, V\rangle$ *over propositional atoms* P *is a relation between worlds and propositional formulas. We define this relation inductively over the structure of a formula. For any* $w \in W$,

$$
\begin{array}{lcr}
w \models_K p & \equiv & p \in V(w) \\
w \models_K \varphi \wedge \psi & \equiv & w \models_K \varphi \text{ and } w \models_K \psi \\
w \models_K \varphi \vee \psi & \equiv & w \models_K \varphi \text{ or } w \models_K \psi \\
w \models_K \sim \varphi & \equiv & \forall w' \in W.\ w\ R\ w' \text{ implies } w' \not\models_K \varphi \\
w \models_K \varphi \supset \psi & \equiv & \forall w' \in W.\ wRw' \text{ and } w' \models_K \varphi \text{ implies } w' \models_K \psi
\end{array}
$$

We say that a formula φ is satisfied at a world w in a structure K if $w \models_K \varphi$.

Definition 3 *Let* $K = \langle W, R, V\rangle$ *be a Kripke structure, then a formula* φ *is* satisfied *in* K *if and only if* $\exists w \in W.\ w \models_K \varphi$. *A formula* φ *is* valid *in* K *if and only if* $\forall w \in W.\ w \models_K \varphi$. *A formula* φ *is* valid *if and only if* φ *is valid in all Kripke structures.*

Intuitionistic logic is sound and complete with respect to Definition 2. However, instead of the completeness relying on a construction of a countermodel for every non-valid formula, there is one *canonical* model in which every non-valid formula is falsified. We go through these details briefly as we need to refer to these details in the interval models for constructible duality.

Definition 4 *Let* $\Gamma \Rightarrow \Delta$ *be an infinite sequent in a logic* L*, then* $\Gamma \Rightarrow \Delta$ *is* complete *if it is underivable in* L *and for all* φ*, a formula of* L*,* $\varphi \in \Gamma \cup \Delta$.

Lemma 5 *Every finite underivable sequent in either intuitionistic logic or* N^- *can be extended to a complete sequent.*

Definition 6 *Let* $\mathcal{K} = \langle W_k, R_k, V_k\rangle$ *denote the canonical refutation model for* I_p*. We have*

1. W_k *as the set of all* $\Gamma \Rightarrow \Delta$ *that are complete infinite sequents,*
2. $\Gamma_i \Rightarrow \Delta_i\ R_k\ \Gamma_j \Rightarrow \Delta_j$ *if and only if* $\Gamma_i \subset \Gamma_j$*, and*
3. $p \in V_k(\Gamma_i \Rightarrow \Delta_i)$ *if and only if* $p \in \Gamma_i$.

Theorem 7 *If* $\Gamma' \Rightarrow \Delta'$ *is a finite underivable sequent in* I_p*, then it can be extended to a complete sequent* $\Gamma \Rightarrow \Delta$ *such that in the canonical refutation model* $\mathcal{K}$*,* $\Gamma \Rightarrow \Delta \not\models_{\mathcal{K}} \Gamma \Rightarrow \Delta$.

7 Interval Kripke Structures for Constructible Falsity

Thomason proposed a Kripke structure interpretation with three truth values for which he shows that N is sound and complete, but not self-dual. The following model is sound and complete and self-dual for N with the caveat the the pair of worlds be proper intervals(Patterson 1996). Here we present the Kripke models for N^- of which constructible duality is a conservative extension.

Definition 8 *Let* $M = \langle W, R, V\rangle$ *be a propositional intuitionistic model over* P*. The dual of* M *is* $M^\perp = \langle W, R^\cup, \bar{V}\rangle$ *where* $R^\cup$ *is the inverse of* R *and* $\forall w \in W.\ \bar{V}(w) = P - V(w)$.

Definition 9 *A model for N^- is a propositional intuitionistic model $M = \langle W, R, V\rangle$ over P with the entailment relation $\models_M$ for* truth *and $\|\!\models_{M^\perp}$ for* falsity *defined by recursion on φ. For any $w_p, w_n \in W$. such that $w_p \neq w_n$.*

$$
\begin{array}{lllr}
p & (w_p, w_n) \models_M p & = & p \in V(w_p) \\
\varphi \wedge \psi & (w_p, w_n) \models_M \varphi \wedge \psi & = & (w_p, w_n) \models_M \varphi \text{ and } (w_p, w_n) \models_M \psi \\
\varphi \vee \psi & (w_p, w_n) \models_M \varphi \vee \psi & = & (w_p, w_n) \models_M \varphi \text{ or } (w_p, w_n) \models_M \psi \\
\varphi \supset \psi & (w_p, w_n) \models_M \varphi \supset \psi & = & \forall w', w'' \in W.\ w' \neq w'' \text{ and } w_p\, R\, w' \text{ and } w''\, R\, w_n \text{ and} \\
 & & & (w', w'') \models_M \varphi \text{ implies } (w', w'') \models_M \psi \\
\neg\varphi & (w_p, w_n) \models_M \neg\varphi & = & (w_n, w_p) \|\!\models_{M^\perp} \varphi \\
p & (w_n, w_p) \|\!\models_{M^\perp} p & = & p \in \bar{V}(w_n) \\
\varphi \wedge \psi & (w_n, w_p) \|\!\models_{M^\perp} \varphi \wedge \psi & = & (w_n, w_p) \|\!\models_{M^\perp} \varphi \text{ or } (w_n, w_p) \|\!\models_{M^\perp} \psi \\
\varphi \vee \psi & (w_n, w_p) \|\!\models_{M^\perp} \varphi \vee \psi & = & (w_n, w_p) \|\!\models_{M^\perp} \varphi \text{ and } (w_n, w_p) \|\!\models_{M^\perp} \psi \\
\varphi \supset \psi & (w_n, w_p) \|\!\models_{M^\perp} \varphi \supset \psi & = & (w_p, w_n) \models_M \varphi \text{ and } (w_n, w_p) \|\!\models_{M^\perp} \psi \\
\neg\varphi & (w_n, w_p) \|\!\models_{M^\perp} \neg\varphi & = & (w_p, w_n) \models_M \varphi
\end{array}
$$

Lemma 10 Monotonicity *Let $M = \langle W, R, V\rangle$ be a propositional intuitionistic model over P. For all $w_p, w', w'', w_n \in W$. such that $w_p\, R\, w' \wedge w''\, R\, w_n$ and $w_p \neq w_n$ and $w' \neq w''$, and for all formulas φ in N^-, $(w_p, w_n) \models_M \varphi$ implies $(w', w'') \models_M \varphi$, and $(w_n, w_p) \|\!\models_{M^\perp} \varphi$ implies $(w'', w') \|\!\models_{M^\perp} \varphi$.*

Theorem 11 *All rules of N^- are sound with respect to Definition 9.*

Our completeness theorem below asserts that any underivable formula is refutable in a canonical model. In fact, the canonical refutation model $\mathcal{K}$ for positive intuitionistic logic I_p is in some sense the canonical refutation model for N^-. Any underivable formula in N^- can be extended to a complete sequent $\Gamma \Rightarrow \Delta$ that can be syntactically partitioned into $\Gamma_p, \Gamma_n, \Gamma_o \Rightarrow \Delta_p, \Delta_n, \Delta_o$ where each φ in $\Gamma_p \cup \Delta_p$ is a formula in I_p, and each φ in $\Gamma_n \cup \Delta_n$ is a negation $\neg$ of an I_p formula. Then $\Gamma_p \Rightarrow \Delta_p$ is a complete sequent for I_p; hence $\Gamma_p \Rightarrow \Delta_p$ is a world in $\mathcal{K}$. The contrapositive of $\Gamma_n \Rightarrow \Delta_n$ is also a world in $\mathcal{K}$. These are the two worlds in $\mathcal{K}$ that falsify $\Gamma \Rightarrow \Delta$.

Definition 12 *We say that a sequent in N^-, $\Gamma \Rightarrow \Delta$ is* falsified *in M if there exists two worlds $w_p, w_n \in W$ $w_p \neq w_n$ and $(w_p, w_n) \models_M \wedge\Gamma$ and $(w_p, w_n) \not\models_M \vee\Delta$.*

Definition 13 *Let Θ be a set of formulas. We define $\Theta^* = \{\varphi \mid \neg\varphi \in \Theta\}$ and $\neg\Theta = \{\neg\varphi \mid \varphi \in \Theta\}$.*

Lemma 14 *If Θ is a set of formulas of N^-, then Θ can be syntactically partitioned into $\Theta_p \cup \Theta_n \cup \Theta_o$ where Θ_p is a set of formulas in I_p, Θ_n is a set of formulas in $\neg I_p$ and Θ_o is disjoint from $I_p \cup \neg I_p$.*

Theorem 15 *Let $M = \langle W, R, V\rangle$ be a propositional model for I_p, and let $\Gamma \Rightarrow \Delta = \Gamma_p, \Gamma_n, \Gamma_o \Rightarrow \Delta_p, \Delta_n, \Delta_o$ be a complete sequent in N^- such that $\Gamma_p \neq \Delta_n^*$. Every $w_p, w_n \in W$. which falsifies $\Gamma_p, \Gamma_n \Rightarrow \Delta_p, \Delta_n$ falsifies $\Gamma \Rightarrow \Delta$.*

Lemma 16 *For any complete sequent $\Gamma_p, \Gamma_n, \Gamma_o \Rightarrow \Delta_p, \Delta_n, \Delta_o$ in N^-, both $\Gamma_p \Rightarrow \Delta_p$ and $\Delta_n^* \Rightarrow \Gamma_n^*$ are complete sequents for I_p.*

Both $\Gamma_p \Rightarrow \Delta_p$ and $\Delta_n^* \Rightarrow \Gamma_n^*$ are worlds in the canonical refutation model $\mathcal{K}$, by Definition 6.

Lemma 17 *For all complete sequents $\Gamma_p, \Gamma_n, \Gamma_o \Rightarrow \Delta_p, \Delta_n, \Delta_o$ in N^- such that $\Gamma_p \neq \Delta_n^*$, the pair of worlds $(\Gamma_p \Rightarrow \Delta_p, \Delta_n^* \Rightarrow \Gamma_n^*)$ falsifies $\Gamma_p, \Gamma_n \Rightarrow \Delta_p, \Delta_n$ in $\mathcal{K}$.*

Theorem 18 Completeness *If $\Gamma' \Rightarrow \Delta'$ is a finite underivable sequent in N^-, then it can be extended to a complete sequent $\Gamma \Rightarrow \Delta$ written equivalently as $\Gamma_p, \Gamma_n, \Gamma_o \Rightarrow \Delta_p, \Delta_n, \Delta_o$, such that the pair of worlds $(\Gamma_p \Rightarrow \Delta_p, \Delta_n^* \Rightarrow \Gamma_n^*)$ falsifies $\Gamma \Rightarrow \Delta$ in $\mathcal{K}$.*

Proof: For all finite underivable sequents $\Gamma' \Rightarrow \Delta'$, they can be extended to complete underivable sequents $\Gamma_p, \Gamma_n, \Gamma_o \Rightarrow \Delta_p, \Delta_n, \Delta_o$ such that $\Gamma_p \neq \Delta_n^*$. By Lemma 17, $(\Gamma_p \Rightarrow \Delta_p, \Delta_n^* \Rightarrow \Gamma_n^*)$ falsifies $\Gamma \Rightarrow \Delta$ in $\mathcal{K}$. This suffices by application of Theorem 15. ∎

8 Substitutability

Usually $\varphi \equiv \psi$ means that φ and ψ are substitutable for each other under all logical operators. This is not the case in N^-. Let $\varphi = \neg(\alpha \supset \beta)$ and let $\psi = \alpha \wedge \neg\beta$, then $\varphi \equiv \psi$, but $\neg\varphi \not\equiv \neg\psi$.

There is a notion of implication which is substitutable under all operators in N^-. We define $A \multimap B = A \supset B \;\wedge\; \neg B \supset \neg A$. This operator makes a substitutable equivalence $A \multimap B \wedge B \multimap A$.

What then should we to use as conjunction which acts with $\multimap$ to give us a deduction rule? We follow linear logic and define $A \otimes B = \neg(A \multimap \neg B)$. $(A \otimes B) \multimap C = A \multimap (B \multimap C)$ in N^- as required. These connectives are definable, and are not equivalent to any other connectives already in N^-.

There is a problem with conjunction $\otimes$ however; it does not have a definable unit. This was the reason for adding to N^- two constants, $\top$ the tensor unit (usually denoted $\mathbf{I}$) and $\perp$ its counterpart. In addition we added the unit for $\vee$, 0, and $\wedge$, 1.

9 Heyting Algebras for Intuitionistic Logic

This section will show that a Heyting algebra (or a pseudo-Boolean algebra) is the algebra for intuitionistic logic. This was shown by McKinsey and Tarski (1946) using a result of Birkhoff (1933) which showed that topologically distributive lattices are pseudo-complemented. This particular presentation follows Fitting (1969).

Definition 19 *In a distributive lattice $\mathcal{X} = (X, \leq, \sqcap, \sqcup, 0)$ with meet, $\sqcap$, join, $\sqcup$, and bottom element 0, the* pseudo-complement *of x* relative *to x', denoted $x \supset x'$ is the element* $\max\{y | x \sqcap y \leq x'\}$. *The pseudo-complement of x relative to 0 is called the* pseudo-complement *and is denoted $-x$.*

Definition 20 *A* pseudo-*Boolean algebra is a relatively pseudo-complemented distributive lattice with a bottom element denoted 0. -0 is denoted 1.*

Definition 21 *Let $\mathcal{X}$ be a pseudo-Boolean algebra and P a set of propositional atoms. Given any function $h: P \rightarrow \mathcal{X}$, we extend h to all formulas φ of intuitionistic logic as follows:*

$$\begin{array}{rclcrcl} h(\alpha \wedge \beta) & = & h(\alpha) \sqcap h(\beta) & \quad & h(\alpha \vee \beta) & = & h(\alpha) \sqcup h(\beta) \\ h(\alpha \supset \beta) & = & h(\alpha) \supset h(\beta) & \quad & h(\sim \alpha) & = & -h(\alpha) \end{array}$$

Definition 22 *A formula φ is* satisfied *in $\mathcal{X}$, a pseudo-Boolean algebra, if and only if $h(\varphi) = 1$. A formula φ is* valid *in $\mathcal{X}$ if and only if for all $h: P \rightarrow \mathcal{X}$, $h(\varphi) = 1$. A formula φ is* valid *if and only if for all pseudo-Boolean algebras $\mathcal{X}$ and for all $h: P \rightarrow \mathcal{X}$, $h(\varphi) = 1$.*

Theorem 23 Soundness *Every derivable sequent in intuitionistic logic is algebraically valid.*

We can prove completeness of the algebraic semantics analogously to using canonical Kripke models by using a canonical valuation map on the free algebra. This method uses H_f the free pseudo-Boolean algebra over P generators for the theory of intuitionistic logic with atoms P. We instead obtain completeness by translating the Kripke models for intuitionistic logic to a pseudo-Boolean algebra.

Definition 24 *Let $\langle W, R, V\rangle$, be a Kripke model. Let $Fil(\langle W, R\rangle)$ denote set of of all R-upward-closed subsets of $\langle W, R\rangle$, that is each $W' \in Fil(\langle W, R\rangle)$ satisfies $(w \in W' \wedge wRw')$ implies $w' \in W'$.*

All R-downward-closed subsets of $\langle W, R\rangle$ are denoted $Idl(\langle W, R\rangle)$, and $Idl(\langle W, R\rangle) \cong Fil(\langle W, R\rangle)^{op}$.

Lemma 25 *$Fil((W, R))$ ordered under subset inclusion is a pseudo-Boolean algebra where the lattice meet is intersection and the lattice join is union.*

The relative pseudo-complement of W_1 relative to W_2, $W_1, W_2 \in Fil(\langle W, R\rangle)$, denoted $W_1 \supset W_2$, is the largest R-upward-closed subset of $(W - W_1) \cup W_2$.

Lemma 26 Beth-Fitting *Let $K = \langle W, R, V\rangle$ be a Kripke model for intuitionistic logic over P. We define $h: P \to Fil(\langle W, R\rangle)$ as $h(p) = \{w \mid p \in V(w)\}$. This assignment extends to all intuitionistic formulas φ such that $h(\varphi) = \{w \mid w \models_K \varphi\}$.*

In particular if φ is valid in the model then it must be true at all worlds and $h(\varphi) = 1 = W$. The backwards construction mapping all proper *prime* filters F over a pseudo-Boolean algebra to Kripke worlds where subset inclusion of filters $F\,R, F' \iff F \subseteq F'$ defines the accessibility relation and where the assignment $p \in V(F) \iff h(p) \in F$ extends to all formula(Fitting 1969).

Corollary 27 Completeness *A formula is derivable in intuitionistic logic* if and only if *it is algebraically valid.*

If $K_B = \langle W, R, V\rangle$ is a **Beth** model for intuitionistic logic, then $H = Fil(\langle W, R\rangle)$ is the pseudo-Boolean algebra representing that model. The backwards construction takes *every* filter in H to be a world in a Beth model (where accessibility and valuation are as above) and provides both an analogue of Stone duality and an instance of the Yoneda embedding.

10 The Algebra of Constructible Duality

Rasiowa proposed a lattice structure for N in (Rasiowa 1974) and an equational definition of an algebra for N, together with a Stone-type duality theorem. However, the basic duality between positive and negative information is brought out and studied by (Vakarelov 1977). Our work can be seen as completing this work by adding the extra operators needed to capture all of $H \times H^{op}$.

Definition 28 *Let $\mathcal{X}$ be a pseudo-Boolean algebra and P a set of propositional atoms. Given any function $h: P \to \mathcal{X} \times \mathcal{X}^{op}$, we extend it to all formula φ of constructible duality as follows: Let $x, x', y, y' \in \mathcal{X}$.*

$$\begin{array}{lclllllcl}
Formulas & & & Units & & & BaseCase & & \\
h(\alpha \wedge \beta) & = & h(\alpha) \sqcap h(\beta) & h(\mathbf{1}) & = & (1,0) & (x,x') \sqcap (y,y') & = & (x \sqcap y, x' \sqcup y') \\
h(\alpha \vee \beta) & = & h(\alpha) \sqcup h(\beta) & h(\mathbf{0}) & = & (0,1) & (x,x') \sqcup (y,y') & = & (x \sqcup y, x' \sqcap y') \\
h(\alpha \supset \beta) & = & h(\alpha) \supset h(\beta) & h(\perp) & = & (0,0) & (x,x') \supset (y,y') & = & (x \supset y, x \sqcap y') \\
h(\neg\alpha) & = & \neg h(\alpha) & h(\top) & = & (1,1) & \neg(x,x') & = & (x', x)
\end{array}$$

Definition 29 *Let $\mathcal{X}$ be a pseudo-Boolean algebra. A formula φ is* valid *if and only if for all pseudo Boolean algebras $\mathcal{X}$ and for all $h: P \to \mathcal{X} \times \mathcal{X}^{op}$, $h(\varphi) = (1, -)$, where $-$ means any valuation.*

Theorem 30 Soundness *Every derivable sequent in constructible duality is valid.*

Again for completeness we could use the canonical evaluation map on the free algebra for constructible duality over P. Whenever H_f is a free pseudo-Boolean algebra over P, then $H_f \times H_f^{op}$ is the free algebra for constructible duality. To show the similarly of the previous technique, we translate our interval Kripke models into our pair algebra.

Lemma 31 Symmetric Beth-Fitting *Let $K = \langle W, R, V \rangle$ be a Kripke structure for constructible duality. If h is any function $h: P \to Fil(\langle W, R \rangle) \times Idl(\langle W, R \rangle)$ such that $h(p) = \{w \mid p \in V(w)\} \times \{w \mid p \notin V(w)\}$, then h extends to all formulas φ such that*
$h(\varphi) = \{w_p \mid \exists w_n'.(w_p, w_n') \models_K \varphi\} \times \{w_n \mid \exists w_p'.(w_n, w_p') \Vdash_{K^\perp} \varphi\}$

The backwards construction mapping all proper *prime* filter-filter pairs over $H \times H^{op}$ to intervals in K, or $K \times K^\perp$, takes prime filter-ideal pairs in H to intervals in K preserving all formulas.

Corollary 32 Completeness *A formula is derivable in constructible duality* if and only if *it is algebraically valid.*

If $K_B = \langle W, R, V \rangle$ is a **Beth** model for intuitionistic logic, then intervals may again be used for a **Beth** model of constructible duality. The forward construction taking K_B to $Fil(\langle W, R \rangle) \times Idl(\langle W, R \rangle)$ again makes an algebra. The backwards construction takes *every* filter-ideal pair to be an interval in a Beth model for constructible duality. This construction provides an instantiation of a symmetric Yoneda embedding.

11 Relationship to Chu, Dialectica and Linear Logic

We can see any pseudo-Boolean as a simple category where objects are points of the lattice and morphisms are $\leq$ in the lattice. The category $\mathcal{X} \times \mathcal{X}^{op}$ has as its objects pairs of objects from $\mathcal{X}$ and has as its morphisms pairs of morphisms from $\mathcal{X}$ such that $(X, X') \in \mathcal{X} \times \mathcal{X}^{op}$ has a morphism to another object (W, W') just when $f: X \to W$ and $g: W' \to X'$ in $\mathcal{X}$ that is whenever $X \leq W$ and $W' \leq X'$.

11.1 Chu

The Chu construction was introduced by Chu, a student of Barr's, in the appendix of (Barr 1979), as a method of producing $*$-autonomous categories.

Definition 33 Chu*(Barr 1979)*
Let $\mathcal{V}$ be a finitely complete symmetric monoidal closed category, and let $\perp$ be an object of $\mathcal{V}$. We define a category $\mathbf{Chu}(\mathcal{V}, \perp)$ *such that*

- *the objects are* Chu spaces*, that is, triples (V, V', v) where V, V' are objects of $\mathcal{V}$ and $v: V \otimes V' \to \perp$ is a morphism in $\mathcal{V}$,*

- *a morphism from (V, V', v) to (W, W', w) is a pair of morphisms $\langle f, g\rangle$ from $\mathcal{V}$, where $f\colon V \to W$ and $g\colon W' \to V'$, satisfying the following* adjunction property

$$\begin{tikzcd} V \otimes W' \arrow[r, "f \otimes 1_{W'}"] \arrow[d, "1_V \otimes g"'] & W \otimes W' \arrow[d, "w"] \\ V \otimes V' \arrow[r, "v"] & \perp \end{tikzcd}$$

Barr shows that for any choice of object, $\perp$, in $\mathcal{V}$, the category $\mathbf{Chu}(\mathcal{V}, \perp)$ is a $*$-autonomous category which embeds $\mathcal{V}$ (Barr 1979). A pseudo-Boolean algebra is a finitely complete symmetric monoidal closed category with product $\wedge$ and residual $\supset$, thus a candidate for the **Chu** construction.

Theorem 34 *Let $\mathcal{X}$ be a pseudo-Boolean algebra.* $\mathbf{Chu}(\mathcal{X}, 1)$ *is isomorphic to* $\mathcal{X} \times \mathcal{X}^{op}$.

Outline of Proof: We construct two functors F from $\mathbf{Chu}(\mathcal{X}, 1)$ to the pair algebra $\mathcal{X} \times \mathcal{X}^{op}$ and $G\colon \mathcal{X} \times X^{op} \to \mathbf{Chu}(\mathcal{X}, 1)$ such that $FG = 1_{\mathcal{X}\times\mathcal{X}^{op}}$ and $GF = 1_{Chu(\mathcal{X},1)}$.

$$\begin{array}{lcl@{\qquad}lcl} F((X, X', x\colon X \wedge X' \to 1)) & = & (X, X') & G((X, X')) & = & (X, X', x\colon X \wedge X' \to 1) \\ F(\langle f, g\rangle) & = & \langle f, g\rangle & G(\langle f, g\rangle) & = & \langle f, g\rangle \end{array}$$

That F is a functor is evident. For G (on objects), x exists and is unique because 1 is the top of $\mathcal{X}$. Thus, we need only check that whenever $\langle f, g\rangle$ is a morphism in $\mathcal{X} \times \mathcal{X}^{op}$, it is a morphism in $\mathbf{Chu}(\mathcal{X}, 1)$. This follows as the adjointness condition below is satisfied whenever f and g exist.

$$\begin{tikzcd} X \wedge W' \arrow[r, "f \wedge 1_{W'}"] \arrow[d, "1_X \wedge g"'] & W \wedge W' \arrow[d, "w"] \\ X \wedge X' \arrow[r, "x"] & 1 \end{tikzcd}$$

The fact that $FG = 1_{\mathcal{X}\times\mathcal{X}^{op}}$ and that $GF = 1_{\mathbf{Chu}(\mathcal{X},1)}$ is immediate. ∎

11.2 Dialectica

The second Dialectica category by de Paiva is usually called $G\mathcal{C}$ (1989) for Girard category. We give an equivalent definition only for G acting upon a pseudo-Boolean algebra.

Definition 35 Dialectica Category for LL
Let $\mathcal{X}$ be a pseudo-Boolean algebra. A category $G\mathcal{X}$ is defined as follows:

- *objects of $G\mathcal{X}$ are morphisms in $\mathcal{X}$ of the form $\alpha\colon A \to V \wedge V'$ and*

- *any morphism of $G\mathcal{X}$ from $\alpha: A \to V \wedge V'$ to $\beta: B \to W \wedge W'$ is a pair $\langle f, g\rangle$ of morphisms from $\mathcal{X}$, $f: V \to W$ and $g: W' \to V'$, such that there exists an $h: B \to A$ in $\mathcal{X}$ such that the following diagram commutes*

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[dr, "\beta"] & A \arrow[r, "\alpha"] & V \wedge V' \arrow[d, "f \wedge 1_{V'}"] \\ & W \wedge W' \arrow[r, "1_W \wedge g"] & W \wedge V' \end{tikzcd}$$

We now consider a smaller category than $G\mathcal{X}$. We consider the category in which all maps come from the same object D rather than from different A's and B's.

Definition 36 *Let $G\mathcal{X}_D$ have as objects the set $D = \{\alpha \mid \alpha: D \to V \wedge V' \in G\mathcal{X} \ \wedge \ V, V' \in \mathcal{X}\}$. Let $G\mathcal{X}_D$ have as morphisms the morphisms in $G\mathcal{X}$ between objects in D.*

This diagram is the restricted morphism condition.

$$\begin{tikzcd} D \arrow[r, "\alpha"] \arrow[d, "\beta"'] & V \wedge V' \arrow[d, "f \wedge 1_{V'}"] \\ W \wedge W' \arrow[r, "1_W \wedge g"] & W \wedge V' \end{tikzcd}$$

Theorem 37 *Let $\mathcal{X}$ be a pseudo-Boolean algebra, $G\mathcal{X}_0$ is isomorphic to $\mathcal{X} \times \mathcal{X}^{op}$.*

Outline of Proof: Again we construct a functor F from $G\mathcal{X}_0$ to the pair algebra and a functor G' from the pair algebra $\mathcal{X} \times \mathcal{X}^{op}$ to $G\mathcal{X}_0$.

$$\begin{array}{lcl@{\qquad}lcl} F(\alpha: 0 \to V \wedge V') & = & (V, V') & G'((V, V')) & = & \alpha: 0 \to V \times V' \\ F(\langle f, g\rangle) & = & \langle f, g\rangle & G'(\langle f, g\rangle) & = & \langle f, g\rangle \end{array}$$

Both F and G' are functors such that $FG' = 1_{\mathcal{X} \times \mathcal{X}^{op}}$ and $G'F = 1_{G\mathcal{X}_0}$. ∎

11.3 Linear Logic

Definition 38 *(Seely) Any $*$-autonomous category with products $\times$, coproducts $+$ together with a comonad $(!, e, \delta)$ such that there are isomorphisms $!(A \times B) \cong !A \otimes !B$, and $!1 \cong \mathbf{I}$ which takes the comonoid structure $(A, !: A \to 1, \Delta: A \to A \times A)$ to the comonoid structure $(A, e: !A \to \mathbf{I}, \delta: !A \to !A \otimes !A)$ is a model of classical linear logic.*

Lemma 39 *(Patterson and Costello 1997)The algebra of constructible duality is a model for classical linear logic.*

Outline of Proof: We have an $\otimes$ with its unit $\top$, a product $\sqcap$ and a coproduct $\sqcup$ in $\mathcal{X} \times \mathcal{X}^{op}$ which is a $*$-autonomous category. The logical operator $!A = A \wedge \top$ generates the needed comonad (which is idempotent) obeying the rules of Seely. ∎

This means there is a sound translation from linear logic to constructible duality and it is given by the table below.

Linear Logic	Constructible Duality
$\mathbf{0}$	$\mathbf{0}$
$\top$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{I}$
$\bot$	$\mathbf{I}$
$A \otimes B$	$\neg((\mathcal{A} \supset \neg\mathcal{B}) \wedge (\mathcal{B} \supset \neg\mathcal{A}))$
$A \multimap B$	$(\mathcal{A} \supset \mathcal{B}) \wedge (\neg\mathcal{B} \supset \neg\mathcal{A})$
$A ⅋ B$	$(\neg\mathcal{A} \supset \mathcal{B}) \wedge (\neg\mathcal{B} \supset \mathcal{A})$
$A \& B$	$\mathcal{A} \wedge \mathcal{B}$
$A \oplus B$	$\mathcal{A} \vee \mathcal{B}$
$A^{\bot}$	$\neg\mathcal{A}$
$!A$	$\mathcal{A} \wedge \mathbf{I}$
$?A$	$\mathcal{A} \vee \mathbf{I}$

where calligraphic letters indicate that the translation is recursively applied to the subformulas.

Theorem 40 *If A is valid in classical linear logic with mix, then $\mathcal{A}$ is valid in constructible duality, where $\mathcal{A}$ is obtained via the translation table above.*

There is a stronger connection however for multiplicative linear logic.

Definition 41 *A formula is balanced if each propositional atom occurs at most twice and if twice with opposite sign.*

Every proof of a formula of multiplicative linear logic is an identification of a proof of a balanced formula.

Theorem 42 *If A is a balanced formula in multiplicative linear logic, then A is valid in multiplicative linear logic with mix, if and only if $\mathcal{A}$ is valid in constructible duality.*

In fact there is only one proof in each case, but that requires a proof of the coherence theorem for $C \times C^{op}$ which is beyond the scope of this text.

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