Exponentials as Projections from Paraconsistent Logics

Anna Patterson and Tom Costello

Computer Science Department, Stanford University, Stanford, CA 94305,

Email {annap,costello}@cs.Stanford.EDU

Abstract

An exponential is a unary operator, reminiscent of a modality. We suggest that exponentials be regarded as logic transformers, transforming *paraconsistent* logics to consistent logics.

Consistent logics may appear as certain syntactic subsets of paraconsistent logics. We consider this embedding algebraically and see that an exponential is both a *projection* and an *interior* operation.

We consider herein a logic *constructible duality* which is a conservative extension of the paraconsistent version of Nelson's constructible falsity. We give an onto representation theorem for the algebra of constructible duality, show that it provides a model for linear logic and in that model ! is an exponential (as defined by linear logicians) a projection operator and an interior operator.

Another exponential ¡ is definable in constructible duality which has no linear logic counterpart. This is also a projection and an interior operator. Closure and interior operations can be thought of as approximations. Thus consistent logics approximate paraconsistent ones.

1 Background

Exponentials were introduced by Girard to recover intuitionistic logic from linear logic (Girard 1987). An exponential is a unary operator, reminiscent of a modality. We suggest that exponentials be regarded as logic transformers, transforming *paraconsistent* logics to consistent logics.

We consider a logic transformer o from logic L to L' to be an operation such that the syntactic subset generated by prefixing all subformulas of each formula of L by o is equivalent to another logic L'.

The most famous example of a logic transformer is the double negation translation of Gödel from intuitionistic logic to classical logic. Gödel showed that prefixing every subformula with $\sim\sim$ embedded classical logic into intuitionistic logic. In the propositional case we can see this as *Glivenko:* A propositional formula φ is valid classically iff $\sim\sim\varphi$ is valid intuitionistically.

In this paper we are most concerned with logic transformers called exponentials. These share certain topological properties with other logic transformers, such as the Gödel translation above.

Before we consider the general properties that these logic transformers have, we consider the logics that we shall translate between. These are classical and intuitionistic logic, two consistent logics, N^- a paraconsistent logic introduced by Nelson (Nelson and Almukdad 1984), certain conservative extensions of N^- that are suggested algebraically, and linear logic, a resource conscious paraconsistent logic introduced by Girard (Girard 1987).

We introduce the algebras of the first four of these logics because all of the logic transformers that we will study have simple algebraic characterizations. We then concentrate on the transformation onto consistent logics from paraconsistent logics. We conclude by describing all of the logic transformers topologically.

2 Logics and their Algebras

We present some logics and their algebras, so that we can later show how certain logic transformers can map one logic into another.

2.1 Intuitionistic Logic

Intuitionistic logic has as connectives $\wedge$, $\vee$, $\supset$ and $\sim$. It can be axiomatized in a Gentzen system as follows. Gentzen sequents $\Gamma \Rightarrow \Delta$ should be read $\wedge\Gamma \supset \vee\Delta$, where Γ, Δ are finite sets of formulas and $\Gamma, \varphi \Rightarrow \Delta, \psi$ is $\wedge\Gamma \wedge \varphi \supset \vee\Delta \vee \psi$ where φ and ψ are formulas of intuitionistic logic.

Axioms

$$(1)\ \Gamma, \varphi \Rightarrow \Delta, \varphi \quad (2)\ \mathbf{0}, \Gamma \Rightarrow \Delta \quad (3)\ \Gamma \Rightarrow \Delta, \mathbf{1}$$

Rules of Inference

$\Rightarrow\wedge$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$	$\wedge\Rightarrow$	$\dfrac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$
$\vee\Rightarrow$	$\dfrac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}$	$\Rightarrow\vee$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$
$\supset\Rightarrow$	$\dfrac{\varphi \supset \psi, \Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \supset \psi, \Gamma \Rightarrow \Delta}$	$\Rightarrow\supset$	$\dfrac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$
$\sim\Rightarrow$	$\dfrac{\sim \varphi, \Gamma \Rightarrow \Delta, \varphi}{\sim \varphi, \Gamma \Rightarrow \Delta}$	$\Rightarrow\sim$	$\dfrac{\varphi, \Gamma \Rightarrow}{\Gamma \Rightarrow \Delta, \sim \varphi}$

We now show that a Heyting algebra is the algebra for intuitionistic logic. This was shown by McKinsey and Tarski (1946) using a result of Birkhoff (1933) which showed that topologically distributive lattices are pseudo complemented. This particular presentation follows Fitting (1969).

Definition 1 *In a distributive lattice $\mathcal{X} = (X, \leq, \sqcap, \sqcup, 0)$ with meet, $\sqcap$, join, $\sqcup$, and bottom element 0, the* pseudo-complement *of x* relative *to x', denoted $x \supset x'$ is the element in X which is* $\max\{y | x \sqcap y \leq x'\}$. *The pseudo-complement of x relative to 0 is called the* pseudo-complement *and is denoted $-x$.*

Definition 2 *A* Heyting *algebra is a relatively pseudo-complemented distributive lattice with a bottom element denoted 0. -0 is denoted 1.*

Proposition 3 *Let $\mathcal{X}$ be a Heyting algebra and P a set of propositional atoms. Given any function $h: P \rightarrow \mathcal{X}$, we can extend h to all formulas φ of intuitionistic logic as follows:*

$$
\begin{array}{lcllcl}
h(\mathbf{0}) & = & 0 & h(\mathbf{1}) & = & 1 \\
h(\alpha \wedge \beta) & = & h(\alpha) \sqcap h(\beta) & h(\alpha \vee \beta) & = & h(\alpha) \sqcup h(\beta) \\
h(\alpha \supset \beta) & = & h(\alpha) \supset h(\beta) & h(\sim \alpha) & = & -h(\alpha)
\end{array}
$$

Definition 4 *A formula φ is* satisfied *in $\mathcal{X}$, a Heyting algebra, if and only if $h(\varphi) = 1$. A formula φ is* algebraically valid in $\mathcal{X}$ *if and only if for all $h: P \rightarrow \mathcal{X}$, $h(\varphi) = 1$. A formula φ is* algebraically valid *if and only if for all Heyting algebras $\mathcal{X}$ and for all $h: P \rightarrow \mathcal{X}$, $h(\varphi) = 1$.*

Theorem 5 Soundness and Completeness *A formula is derivable in intuitionistic logic if and only if it is algebraically valid.*

Another very useful view of a Heyting algebra is as a topology.

Definition 6 *A* topological space $\mathcal{T}_{X_\mathcal{O}}$ *is a pair* $(X, \mathcal{O})$ *consisting of a set* X *together with a set of subsets of* X *which are* open *sets.* $\mathcal{O}$ *is a system of* open *sets when any union of open sets is open, the intersection of two open sets is open, and* $\emptyset$ *and* X *are open.*

The interior, *Int*, of a subset Y of X is the largest open set contained in Y. The closure of a subset Y is the smallest set that contains Y and is the complement of an open set. We say an open set in $\mathcal{O}$ is *regular* in case it coincides with the interior of its own closure. We write the complement of a set Q as Q^- and the closure as Q'. We write $Q^{-\prime}$ as $Q^\perp$, thus regular open sets obey $Q = Q^{\perp\perp}$.

Proposition 7 *The open sets* $\mathcal{O}$ *of a non-empty topological space* $\mathcal{T}_{X_\mathcal{O}}$ *form a Heyting algebra with respect to the Heyting operations defined by*

Heyting		*Topology*	*Heyting*		*Topology*
0	$=$	$\emptyset$	1	$=$	X
$P \sqcap Q$	$=$	$P \cap Q$	$P \sqcup Q$	$=$	$P \cup Q$
$-P$	$=$	$\mathrm{Int}(P^-)$	$P \supset Q$	$=$	$\mathrm{Int}(P^- \cup Q)$

2.2 Classical Logic

Classical propositional logic has the connectives $\wedge$, $\vee$, $\sim$. It can be axiomatized by adding the axiom scheme $\Gamma \Rightarrow \Delta, \sim \phi \vee \phi$ to the intuitionistic rules above. Its algebras are Boolean algebras. The most normal example of a Boolean algebra is the set of all subsets of a given set. We will find it more useful to consider the following algebra.

Proposition 8 *The regular open sets of a non-empty topological space* $\mathcal{T}_{X_\mathcal{O}}$ *form a Boolean algebra with respect to the distinguished Boolean elements and operations defined by*

Boolean		*Topology*	*Boolean*		*Topology*
$\mathbf{0}$	$=$	$\emptyset$	$\mathbf{1}$	$=$	X
$P \wedge Q$	$=$	$P \cap Q$	$P \vee Q$	$=$	$(P \cup Q)^{\perp\perp}$
$\neg P$	$=$	$P^\perp$			

Therefore, the double negation $\sim\sim$ translation of Gödel can be seen as collapsing a topology onto its regular open sets.

2.3 Constructible Falsity

The logic of constructible falsity, N (Nelson 1949, Nelson and Almukdad 1984), adds an involutory negation $\neg$ to positive intuitionistic logic. This negation asserts the existence of a counterexample to a formula, not merely that the truth of a formula implies contradiction. N contains the positive intuitionistic connectives $\wedge$, $\vee$, $\supset$ as well as the new involutory negation, $\neg$. We consider its paraconsistent version N_c^- which drops the axiom $\Gamma, \varphi, \neg\varphi \Rightarrow \Delta$ from N but keeps the constants for falsity $\mathbf{0}$ and truth $\mathbf{1}$.

2.3.1 Adding a Constructive Negation

We can axiomatize N_c^- by adding the following rules to positive intuitionistic logic with constants. This makes N_c^- conservative extension of intuitionistic logic because intuitionistic negation is definable as $\sim \varphi = \varphi \supset \mathbf{0}$ and satisfies the axiom scheme $\Gamma, \varphi, \sim \varphi \Rightarrow \Delta$.

$$\textbf{Axioms} \quad (4)\Gamma, \neg\mathbf{1} \Rightarrow \Delta \quad (5)\Gamma \Rightarrow \Delta, \neg\mathbf{0}$$

$$\neg\neg \Rightarrow \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\neg\neg\varphi, \Gamma \Rightarrow \Delta} \qquad \Rightarrow \neg\neg \quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi}$$

$$\neg\wedge \Rightarrow \quad \frac{\neg\varphi, \Gamma \Rightarrow \Delta \quad \neg\psi, \Gamma \Rightarrow \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta} \qquad \Rightarrow \neg\wedge \quad \frac{\Gamma \Rightarrow \Delta, \neg\varphi, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \wedge \psi)}$$

$$\neg \supset\Rightarrow \quad \frac{\varphi, \neg\psi, \Gamma \Rightarrow \Delta}{\neg(\varphi \supset \psi), \Gamma \Rightarrow \Delta} \qquad \Rightarrow \neg \supset \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \supset \psi)}$$

Note $\varphi \vee \psi$ is definable as $\neg(\neg\varphi \wedge \neg\psi)$.

We note that the law of the excluded middle does not hold for this new negation even though it obeys the deMorgan laws and involution. However, we pay a cost for the addition of this new negation. It breaks a notion found in standard logics which we call the *substitutability* of implication.

2.3.2 Adding operators to N_c^-

In contrast with intuitionistic and classical logic the implication of N_c^- is not substitutable. That is, interderivablity does not form a congruence. This means that two formulas can be equivalent in one context and not in another.

We consider an example of this phenomena: $\neg(A \supset B) \equiv A \wedge \neg B$ where $A \equiv B$ when $A \supset B$ and $B \supset A$.

This equivalence holds but the negation of each of these formulas is not equivalent. $\neg\neg(A \supset B) \not\equiv \neg(A \wedge \neg B)$ because using the involution of negation on the left and the deMorgan equivalence on the right we get the following inequality $A \supset B \not\equiv \neg A \vee B$. If this were an equivalence, this logic would be classical.

This causes difficulty in forming an algebra because the standard method of creating points in an algebra is to assign to each point in the algebra an equivalence class of interderivable formulae. That is a point in the algebra is a formula A and all other B for which $A \equiv B$. We denote the equivalence class of a formula A by $[A]$. The ordering in the algebra $\leq$ is an ordering on points where $[A] \leq [B]$, if and only if $A \supset B$ is a valid formula, and thus $[A] < [B]$ if $A \not\equiv B$ but $A \supset B$.

To ameliorate this problem of a substitutable implication for N, $A \multimap B = A \supset B \wedge \neg B \supset \neg A$ was introduced by (Rasiowa 1974, Vakarelov 1977) where it was denoted $\Rightarrow$. This new implication is substitutable and forms a congruence on equivalence class of interderivable formulae. That is, $[A]$ is the equivalence class of formulas B such that $A \multimapboth B$ and if C is derivable and contains an occurrence of A then $C[B/A]$ is derivable (where $C[B/A]$ denotes replacing every occurrence of A by B and $A \multimapboth B = A \multimap B \wedge B \multimap A$).

This substitutable implication forms the ordering on the algebras considered by (Rasiowa 1974) because $[A] \leq [B]$ in the algebra for N, if and only if $A \multimap B$ is a valid formula in N, and thus $[A] < [B]$ if $A \not\multimapboth B$ but $A \multimap B$.

Now that we have an "ordering", $\multimap$ on the points of an algebra, we would like some sort of conjunction with respect to which we may have a deduction theorem in the algebraic setting.

We now introduce a *new*, but definable, operation, tensor, $A \otimes B$ defined as $\neg(A \multimap \neg B)$. This new operator obeys the rule,

$$(A \otimes B) \multimap C = A \multimap (B \multimap C).$$

Thus it acts much like a conjunction for $\multimap$ and gives us our deduction rule. It turns out that in linear logic $A \otimes B = (A \multimap B^{\perp})^{\perp}$, thus our $\neg$ is acting like $^{\perp}$ in this instance.

This new operator $\otimes$ acts as the conjunction for the overall ordering of the algebra. Even though $\otimes$ is definable, this operator and its interaction with $\multimap$ has not been studied. In this paper, $\otimes$ will be of pivotal importance in our introduction of exponentials to N_c^-.

First however we also define the deMorgan dual of $\otimes$ which we denote $\oplus$. Since, we have an involutory negation $\otimes$ must have a deMorgan dual. This deMorgan dual defines a disjunction associated with $\multimap$. It is called or co-tensor (written $⅋$ in linear logic) as $A \oplus B = \neg(\neg A \otimes \neg B) = \neg A \multimap B$, as the De Morgan dual of $\otimes$. In linear logic $A ⅋ B = A^\perp \multimap B$, thus again our $\neg$ is acting like $^\perp$.

2.3.3 Adding *overdefined*

Once we have added our conjunction $\otimes$ with respect to which the algebraic ordering $\multimap$ is the residual, it is natural to ask whether $\otimes$ has a definable unit?
It does not.

To get the unit for $\otimes$, we must add a new constant $\mathbf{I}$, to N_c^-. This unit is distinct from the other constants, falsity, $\mathbf{0}$, and truth, $\mathbf{1}$. $\mathbf{I}$ is not definable in N_c^-. We can axiomatize this unit as $A \supset \mathbf{I}$, $\neg\mathbf{I} \equiv \mathbf{I}$ where A is any formula in N_c^-.

The addition of $\mathbf{I}$ demands paraconsistency. We have a unit which is equal to its own negation but does not imply everything. Furthermore, this unit has an interpretation as maximally overdefined — both "true" and "false". The negation of overdefined, $\mathbf{I}$, is itself, and every formula implies it. Thus, we could not add $\mathbf{I}$ to a logic containing the axiom $A \wedge \neg A \supset B$ without trivializing the logic.

The sequent rules for this new constant are

Axiom

$$(6)\Gamma \Rightarrow \Delta, \mathbf{I}$$

$$\neg\mathbf{I} \Rightarrow \quad \frac{\mathbf{I}, \Gamma \Rightarrow \Delta}{\neg\mathbf{I}, \Gamma \Rightarrow \Delta} \quad \Rightarrow \neg\mathbf{I} \quad \frac{\Gamma \Rightarrow \Delta, \mathbf{I}}{\Gamma \Rightarrow \Delta, \neg\mathbf{I}}$$

Now that we have added this new constant *overdefined*, it makes sense to add its opposite, *undefined*, written $\perp$ in (Vakarelov 1977).

2.3.4 Adding *undefined*

This constant captures the notion of a proposition that is neither true nor false, i.e. it is *undefined*. As such, $\perp$ implies every formula and like $\mathbf{I}$ it is its own negation. However, it is distinct from **I**, and the constants **1** and **0**. The sequent rules for this new constant are

Axiom

$$(7)\Gamma, \perp \Rightarrow \Delta$$

$$\neg\perp \Rightarrow \quad \frac{\perp, \Gamma \Rightarrow \Delta}{\neg\perp, \Gamma \Rightarrow \Delta} \quad \Rightarrow \neg\perp \quad \frac{\Gamma \Rightarrow \Delta, \perp}{\Gamma \Rightarrow \Delta, \neg\perp}$$

2.3.5 Completing the algebra $H \times H^{op}$

The addition of the two constants $\perp$ and **I** complete N_c^-, in the sense that there is an onto representation theorem for N_c^- with these additional constants with respect to $H \times H^{op}$ where H is a Heyting algebra. This naturally completes N_c^- so that every point in the algebra $H \times H^{op}$ is representable as an equivalence class of interderivable formulas in $N_c^- + \mathbf{I} + \perp$. As $N_c^- + \mathbf{I} + \perp$ is a cumbersome name, we call this constructible duality since every formula has a dual and the logic allows us to construct it.

Before we consider this new algebra for constructible duality, we briefly look at some earlier algebras proposed for variants of N.

Rasiowa proposed a lattice structure for N in (Rasiowa 1974) and an equational definition of an algebra for N, together with a Stone-type duality theorem. However, the basic duality between positive and negative information is more naturally brought out and studied by (Vakarelov 1977). The major difference between his algebra and ours is that we consider $H \times H^{op}$ rather than $H \times H$. That is, we consider that the part of the algebra that captures falsity is order reversed. This corresponds to that fact that falsity propagates in the opposite direction to truth, statements that have more truth naturally have less falsity.

Lemma 9 Representation *Let $\mathcal{X}$ be a Heyting algebra and P a set of propositional atoms. Given any function $h\colon P \to \mathcal{X} \times \mathcal{X}^{op}$, we extend it to all formula φ of constructible duality as follows: Let $x, x', y, y' \in \mathcal{X}$.*

$Formulas$			$Units$			$BaseCase$		
$h(\alpha \wedge \beta)$	$=$	$h(\alpha) \sqcap h(\beta)$	$h(\mathbf{1})$	$=$	$(1,0)$	$(x,x') \sqcap (y,y')$	$=$	$(x \sqcap y, x' \sqcup y')$
$h(\alpha \vee \beta)$	$=$	$h(\alpha) \sqcup h(\beta)$	$h(\mathbf{0})$	$=$	$(0,1)$	$(x,x') \sqcup (y,y')$	$=$	$(x \sqcup y, x' \sqcap y')$
$h(\alpha \supset \beta)$	$=$	$h(\alpha) \supset h(\beta)$	$h(\perp)$	$=$	$(0,0)$	$(x,x') \supset (y,y')$	$=$	$(x \supset y, x \sqcap y')$
$h(\neg\alpha)$	$=$	$\neg h(\alpha)$	$h(\mathbf{I})$	$=$	$(1,1)$	$\neg(x,x')$	$=$	(x', x)

Definition 10 *Let $\mathcal{X}$ be a Heyting algebra. A formula φ is* algebraically valid *if and only if for all Heyting algebras $\mathcal{X}$ and for all $h\colon P \to \mathcal{X} \times \mathcal{X}^{op}$, $h(\varphi) = (1, -)$, where $-$ means any valuation.*

Theorem 11 Soundness and completeness *A formula is derivable in constructible duality* if and only if *it is algebraically valid.*

For completeness we use the canonical evaluation map on the free algebra for constructible duality generated by P. Whenever H_f is a free Heyting algebra with P as its set of generators, then $H_f \times H_f^{op}$ is the free algebra for constructible duality over generators P.

We now have a algebra for N_c^- with our additional units. This algebra bears a strong resemblance to the algebra of intuitionistic logic, in fact, it is two copies of a Heyting algebra, one order reversed, producted together. Thus it clearly contains two copies of intuitionistic logic (or a copy of intuitionistic logic and a copy of Browerian logic (Stone 1937)).

We now show how these two copies can be extracted from constructible duality. Then we consider Girard's linear logic and in particular we discuss its exponentials which will be the tools that carry out our promised decomposition of N_c^-. These exponentials allow us to provide a one-line proof that the representation lemma is onto.

3 A new exponential

We now introduce an operator satisfying all of the rules of an exponential in linear logic which is definable in constructible duality. The definitions and lemmas needed to prove these statements appear in the next section.

This new exponential, !, is defined as $!A = A \wedge \mathbf{I}$. This operation removes the negative content from A. This is done by making the falsity of A true. Therefore, everything implies $\neg !A$. We can see this as $\vdash \phi \supset \neg !A$.

$\vdash \phi \supset \neg !A$	We get
$\vdash \phi \supset \neg(A \wedge \mathbf{I})$	from expanding the definition
$\vdash \phi \supset \neg A \vee \neg \mathbf{I}$	and this using deMorgan. Finally
$\vdash \phi \supset \neg A \vee \mathbf{I}$	follows from the
$\vdash \phi \supset \mathbf{I}$	axiom.

Intuitionistic implication is definable as $!A \multimap B$ using this new exponential and substitutable implication.

$$
\begin{array}{lcll}
A \supset B & = & !A \multimap B & \\
A \supset B & = & A \wedge \mathbf{I} \multimap B & \\
A \supset B & = & ((A \wedge \mathbf{I}) \supset B) \wedge \;\; \neg B \supset \neg(A \wedge \mathbf{I}) & \\
A \supset B & = & ((A \wedge \mathbf{I}) \supset B) \wedge \;\; \neg B \supset (\neg A \vee \neg \mathbf{I}) & \\
A \supset B & = & ((A \wedge \mathbf{I}) \supset B) \wedge \;\; \neg B \supset (\neg A \vee \mathbf{I}) & \\
A \supset B & = & ((A \wedge \mathbf{I}) \supset B) & \quad \mathbf{I} \text{ on the left acts like the unit for } \wedge \\
A \supset B & = & A \supset B &
\end{array}
$$

This new exponential obeys two important rules, $!A \multimap A$, that is, applying ! weakens its argument, and $!!A = !A$, applying ! twice is the same as applying it once. We prove these two properties formally.

Lemma 12 1. $!A \multimap A$ 2. $!!A = !A$

Proof:

1.
$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A, \mathbf{I} \Rightarrow A}{A \wedge \mathbf{I} \Rightarrow A}}{!A \Rightarrow A}}{\Rightarrow !A \supset A} \qquad \dfrac{\dfrac{\dfrac{\neg A \Rightarrow \neg A, \neg \mathbf{I}}{\neg A \Rightarrow \neg(A \wedge \mathbf{I})}}{\neg A \Rightarrow \neg !A}}{\Rightarrow \neg A \supset \neg !A}}{\Rightarrow !A \supset A \wedge \neg A \supset \neg !A}}{\Rightarrow !A \multimap A}
$$

2.
$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A, \mathbf{I} \Rightarrow A}{A \wedge \mathbf{I} \Rightarrow A} \quad \Rightarrow \mathbf{I}}{A \wedge \mathbf{I} \Rightarrow A \wedge \mathbf{I}}}{A \wedge \mathbf{I} \Rightarrow (!A)} \qquad A \wedge \mathbf{I} \Rightarrow \mathbf{I}}{A \wedge \mathbf{I} \Rightarrow (!A) \wedge \mathbf{I}}}{A \wedge \mathbf{I} \Rightarrow !!A}}{!A \Rightarrow !!A}}{\Rightarrow !A \supset !!A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\neg !!A \Rightarrow \neg A, \mathbf{I}}{\neg !!A \Rightarrow \neg A, \neg \mathbf{I}}}{\neg !!A \Rightarrow \neg(A \wedge \mathbf{I})}}{\neg !!A \Rightarrow \neg !A}}{\Rightarrow \neg !!A \supset \neg !A}}{\Rightarrow !A \multimap !!A}
$$
and 1. give us $!A = !!A$

∎

Linear logic contains two exponentials, $!_l$ and $?_l$, which are DeMorgan duals. We also gain a dual exponential ? to our new ! defined as $?A =$

$\neg!\neg A = A \vee \mathbf{I}$.

$$\begin{array}{lcl} \neg!\neg A & = & A \vee \mathbf{I} \\ \neg(\neg A \wedge \mathbf{I}) & = & A \vee \mathbf{I} \\ \neg\neg A \vee \neg \mathbf{I} & = & A \vee \mathbf{I} \\ A \vee \mathbf{I} & = & A \vee \mathbf{I} \end{array}$$

This new exponential obeys three important rules, $A \multimap ?A$, $?A = ??A$ and $A \multimap ?B = \neg B \supset \neg A$. Therefore, ? the deMorgan dual to ! extracts out the dual proof. Remember

$$\begin{array}{lcccc} & A \multimap B & = & A \supset B \;\; \wedge & \neg B \supset \neg A \\ \text{and} & !A \multimap B & = & A \supset B & \\ \text{whereas} & A \multimap ?B & = & & \neg B \supset \neg A. \end{array}$$

Therefore, the ! exponential extracts out the construction of a proof from a proof (in its BHK interpretation (Heyting 1966)) and the dual exponential ? extracts out the construction of a counterexample from the construction of a counterexample.

We now consider how these new exponentials behave algebraically. We show that ! maps $H \times H^{op}$ to $H \times 1$ which is isomorphic to H. This gives algebraic justification to our previously stated intuition that ! extracts the positive information. This means that ! is a projection from the algebra of constructible duality to a Heyting algebra; thus ! projects a paraconsistent logic to a consistent logic i.e. intuitionistic logic.

Similarly ? maps $H \times H^{op}$ to $1 \times H^{op}$ which is isomorphic to H^{op}. This gives algebraic justification that ? extracts the negative information. This means that ? is a projection from the algebra of constructible duality to a dual-Heyting algebra (also called a Browerian algebra); thus ? projects the logic of constructible duality to Browerian logic (McKinsey and Tarski 1944, Birkhoff 1948).

Proposition 13 *! maps $H \times H^{op}$ to $H \times 1$ and ? maps $H \times H^{op}$ to $1 \times H^{op}$.*

Proof: Let A be a point $A = \langle a_1, a_2 \rangle$ in $H \times H^{op}$. Then, by definition, $!A = A \wedge \mathbf{I}$ where $\mathbf{I} = \langle 1, 1 \rangle$. Therefore $!A = \langle a_1, a_2 \rangle \wedge \langle 1, 1 \rangle$ by the definition of $\wedge$ gives, $!A = \langle a_1 \sqcap 1, a_2 \sqcup 1 \rangle$. Since 1 is the top of the lattice, this gives, $!A = \langle a_1, 1 \rangle$. Therefore, every point $\langle a_1, a_2 \rangle$ is mapped to $\langle a_1, 1 \rangle$, as required. The ? case is similar.

Thus these new exponentials project out an intuitionistic algebra from the larger algebra of constructible duality. To project to classical logic we simply perform the ! translation followed by the $\sim\sim$ closure. Before we show that these exponentials are the same exponentials as linear logic, we consider two more exponentials. These project in another way. They give $H \times 0$ and $0 \times H^{op}$. As ! and ? came from the overdefined unit, these new exponentials come from the unit for undefined.

4 The *undefined* exponential

More exponentials can be added to N_c^- using the constant for *undefined*, $\perp$. A weak exponential ¡ is definable as $¡A = A \wedge \perp$.

This weak exponential obeys $¡A \multimap A$ and $¡¡A = A$ and has its deMorgan dual $¿A = A \vee \perp$ where $A \multimap ¿A$ and $¿A = ¿¿A$. This deMorgan dual ¿ of ¡ also extracts intuitionistic implication from substitutable implication $\multimap$. However, it is applied to the second argument via the rule $A \multimap ¿B$.

$$
\begin{array}{lcll}
A \supset B & = & A \multimap ¿B & \\
A \supset B & = & A \multimap (B \vee \perp) & \\
A \supset B & = & (A \supset (B \vee \perp)) \wedge & \neg(B \vee \perp) \supset \neg A \\
A \supset B & = & (A \supset (B \vee \perp)) \wedge & (\neg B \wedge \neg\perp) \supset \neg A \\
A \supset B & = & (A \supset B) \wedge & (\neg B \wedge \perp) \supset \neg A \\
A \supset B & = & A \supset B &
\end{array}
$$

And ¡ extracts out the dual intuitionistic proof that is $¡A \multimap B = \neg B \supset \neg A$. In summary:

$$
\begin{array}{llclcl}
 & A \multimap B & \equiv & A \supset B & \wedge & \neg B \supset \neg A \\
\text{and} & !A \multimap B & \equiv & A \supset B & & \\
\text{and} & A \multimap ¿B & \equiv & A \supset B & & \\
\text{whereas} & A \multimap ?B & \equiv & & & \neg B \supset \neg A \\
\text{and} & ¡A \multimap B & \equiv & & & \neg B \supset \neg A
\end{array}
$$

All four exponentials are distinct. It is notable that $A \multimap ¿\, B$ is not strongly equivalent, i.e. $\circ\!\!-\!\!\circ$, to $!A \multimap B$ because $\neg(A \multimap ¿\, B)$ is not equivalent to $\neg(\,!A \multimap B)$.

Proposition 14 *¡ maps $H \times H^{op}$ to $0 \times H^{op}$ and ¿ maps $H \times H^{op}$ to $H \times 0$.*

Theorem 15 *Given any Heyting algebra, H and any point $\langle h, h' \rangle$ in $H \times H^{op}$ there is a formula in constructible duality which represents $\langle h, h' \rangle$.*

Proof: Pick two points $\langle h, b\rangle$ and $\langle c, h'\rangle$ of $H \times H^{op}$. We construct $\langle h, h'\rangle$ via $!\langle h, b\rangle \vee ¡\langle c, h'\rangle$ — this expands to $\langle h, 1\rangle \vee \langle 0, h'\rangle$ and then to $\langle h, h'\rangle$.

Now we use the onto representation theorem from intuitionistic logic to Heyting algebras and the onto representation theorem from Browerian logic to dual Heyting algebras to conclude that with the simple construction above every pair of elements in $H \times H^{op}$ is constructible. ∎

We can put this together with the representation lemma to conclude that constructible duality has an onto representation theorem with respect to $H \times H^{op}$.

We have shown that these new operators act as projections. We now proceed to establish the claim that these projection operations are exponentials, we need to show that they satisfy those requirements. Exponentials were introduced by Girard in linear logic so for a definition of an exponential we need to review their properties in the linear logic setting.

5 Linear Logic

Linear logic is a paraconsistent resource conscious logic introduced by Girard (Girard 1987). It has as five binary operators $\multimap, \otimes, ⅋, \&, \oplus$, a unary $^\perp$ and two unary exponentials $!_l$ and $?_l$ where $!_l A = (?_l A^\perp)^\perp$. It has four units $1, 0, T, \perp$. It can be axiomatized in a Gentzen system as follows.

Bold letters denote lists of formulas. Propositions are constructed from atomic ones (letters and their negations) by applying the binary connectives, $\otimes$, ⅋, $\otimes$, & where the linear negation of a proposition is defined as follows:

$$1^\perp = \perp, \quad \perp^\perp = 1 \quad T^\perp = 0, \quad 0^\perp = T$$
$$(a^\perp)^\perp = a$$
$$(A \otimes B)^\perp = A^\perp ⅋ B^\perp, (A ⅋ B)^\perp = A^\perp \otimes B^\perp,$$
$$(A \oplus B)^\perp = A^\perp \& B^\perp, (A \& B)^\perp = A^\perp \oplus B^\perp,$$

Logical Axioms

$\vdash A, A^{\perp}$

Cut rule:

$$\frac{\vdash \mathbf{B}, A \quad \vdash, A^{\perp}, \mathbf{C}}{\vdash \mathbf{B}, \mathbf{C}}\ \mathrm{Cut}.$$

Exchange rule:

$$\frac{\vdash \mathbf{A}}{\vdash \mathbf{B}}\ \mathrm{EXCH},$$

$\mathbf{B}$ is a permutation of $\mathbf{A}$.

Additive rules:

$\vdash T, \mathbf{A}$

$$\frac{\mathbf{C}, A \quad \vdash \mathbf{C}, B}{\vdash \mathbf{C}, A \& B}\&1, \frac{A, \mathbf{C}}{\mathbf{D}, A \oplus B, \mathbf{C}}1\oplus \frac{B, \mathbf{C}}{\mathbf{D}, A \oplus B, \mathbf{C}}2\oplus.$$

Multiplicative rules:

$\vdash \mathbf{1}(\text{axiom}),\quad \dfrac{\vdash \mathbf{A}}{\vdash \perp, \mathbf{A}}\perp,$

$$\frac{\vdash A, \mathbf{C} \quad \vdash B, \mathbf{D}}{\vdash A \otimes B, \mathbf{C}, \mathbf{D}}\otimes, \frac{\vdash \mathbf{D}, A, B, \mathbf{C}}{\vdash \mathbf{D}, A \parr B, \mathbf{C}}$$

Exponential rules:

$$\frac{\vdash A, \mathbf{B}}{\vdash ?_l A, \mathbf{B}}D?_l \quad (\text{ dereliction}), \quad \frac{\vdash ?_l A, ?_l A, \mathbf{B}}{\vdash ?_l A, \mathbf{B}}C?_l \quad (\text{contraction}).$$

$$\frac{\vdash \mathbf{B}}{\vdash ?_l A, \mathbf{B}}W?_l \quad (\text{ weakening}), \quad \frac{\vdash A, ?_l \mathbf{B}}{\vdash !_l A, ?_l \mathbf{B}}$$

Rather than give an algebra for linear logic it is more normal to consider categorical models. These models carry with them a notion of proof identity as well as the usual ordering relation of an algebra. We shall not be interested in the conditions of proof identity in this paper.

We consider a theorem by Seely, that provides a definition of a model for linear logic. We then consider how we can instantiate this definition with a structure that we have already considered — $H \times H^{op}$. We need $H \times H^{op}$ to satisfy the prerequisites.

Theorem 16 *(Seely) Any $*$-autonomous category with products $\times$, coproducts $+$ together with a comonad $(!, e, \delta)$ such that there are isomorphisms $!(A \times B) \cong !A \otimes !B$, and $!1 \cong \mathbf{I}$ which takes the comonoid structure $(A, !\colon A \to 1, \Delta\colon A \to A \times A)$ to the comonoid structure $(A, e\colon !A \to \mathbf{I}, \delta\colon !A \to !A \otimes !A)$ is a model of classical linear logic.*

We can see any Heyting algebra as a simple category where objects are points of the lattice and morphisms are $\leq$ in the lattice. The category $\mathcal{X} \times \mathcal{X}^{op}$ has as its objects pairs of objects from $\mathcal{X}$ and has as its morphisms pairs of morphisms from $\mathcal{X}$ such that $(X, X') \in \mathcal{X} \times \mathcal{X}^{op}$ has a morphism to another object (W, W') just when $(X, X'') \multimap (W, W')$ that is when and $X \leq W$ and $W' \leq X'$ in $\mathcal{X}$. We now show that this $H \times H^{op}$ is a $*$-autonomous category. We show this via the Chu construction.

The Chu construction was introduced by Chu, a student of Barr's, in the appendix of (Barr 1979), as a method of producing $*$-autonomous categories.

Definition 17 Chu *(Barr 1979)*
Let $\mathcal{V}$ be a finitely complete symmetric monoidal closed category, and let $\perp$ be an object of $\mathcal{V}$. We define a category $\mathbf{Chu}(\mathcal{V}, \perp)$ *such that*

- *the objects are* Chu spaces*, that is, triples (V, V', v) where V, V' are objects of $\mathcal{V}$ and $v: V \otimes V' \to \perp$ is a morphism in $\mathcal{V}$,*
- *a morphism from (V, V', v) to (W, W', w) is a pair of morphisms $\langle f, g\rangle$ from $\mathcal{V}$, where $f: V \to W$ and $g: W' \to V'$, satisfying the following* adjunction property

$$\begin{array}{ccc} V \otimes W' & \xrightarrow{f \otimes 1_{W'}} & W \otimes W' \\ {\scriptstyle 1_V \otimes g}\downarrow & & \downarrow{\scriptstyle w} \\ V \otimes V' & \xrightarrow{v} & \perp \end{array}$$

Barr shows that for any choice of object, $\perp$, in $\mathcal{V}$, the category $\mathbf{Chu}(\mathcal{V}, \perp)$ is a $*$-autonomous category which embeds $\mathcal{V}$ (Barr 1979). A Heyting algebra is a finitely complete symmetric monoidal closed category with product $\wedge$ and residual $\supset$, thus a candidate for the **Chu** construction.

Theorem 18 *Let $\mathcal{X}$ be a Heyting algebra.* $\mathbf{Chu}(\mathcal{X}, 1)$ *is isomorphic to $\mathcal{X} \times \mathcal{X}^{op}$.*

Outline of Proof: We construct two functors F from $\mathbf{Chu}(\mathcal{X}, 1)$ to the pair algebra $\mathcal{X} \times \mathcal{X}^{op}$ and $G: \mathcal{X} \times X^{op} \to \mathbf{Chu}(\mathcal{X}, 1)$ such that $FG = 1_{\mathcal{X} \times \mathcal{X}^{op}}$ and $GF = 1_{Chu(\mathcal{X},1)}$.

$$
\begin{array}{lcl lcl}
F((X,X',x{:}\,X\wedge X'\to 1)) & = & (X,X') & G((X,X')) & = & (X,X',x{:}\,X\wedge X'\to 1)\\
F(\langle f,g\rangle) & = & \langle f,g\rangle & G(\langle f,g\rangle) & = & \langle f,g\rangle
\end{array}
$$

The fact that $FG = 1_{\mathcal{X}\times\mathcal{X}^{op}}$ and that $GF = 1_{\mathbf{Chu}(\mathcal{X},1)}$ is immediate because $x{:}\,X \wedge X' \to 1$ always exists and is unique in any Heyting algebra. ∎

Thus we have shown that the algebra $H \times H^{op}$ is a $*$-autonomous category.

To instantiate $H \times H^{op}$ in Seely's theorem, we need a comonad $(!, e, \delta)$ such that there are isomorphisms $!(A \times B) \cong !A \otimes !B$, and $!1 \cong \mathbf{I}$ which takes the comonoid structure $(A, !{:}\,A \to 1, \Delta{:}\,A \to A \times A)$ to the comonoid structure $(A, e{:}\,!A \to \mathbf{I}, \delta{:}\,!A \to !A \otimes !A)$

Definition 19 *A co-monad $\langle !, \epsilon{:}\,! \to id, \delta{:}\,! \to !^2\rangle$ consists of an endofunctor $!{:}\,C \to C$ and two natural transformations $\epsilon{:}\,! \to id_C$ and $\delta{:}\,! \to !^2$ such that the diagrams commute*

$$
\begin{tikzcd}
! \arrow[r, "\delta"] \arrow[d, "\delta"'] & !^2 \arrow[d, "!\delta"] \\
!^2 \arrow[r, "\delta_!"] & !^3
\end{tikzcd}
\qquad
\begin{tikzcd}
& ! \arrow[dl, "id"'] \arrow[d, "\delta"] \arrow[dr, "id"] & \\
! & !^2 \arrow[l, "!\epsilon"'] \arrow[r, "\epsilon_!"] & !
\end{tikzcd}
$$

We now show that the operator !, that maps $H \times H^{op}$ to itself, via the rule $!A = A \wedge \mathbf{I}$ gives rise to a co-monad.

Definition 20 *A* closure *operation on some partial order $(X, \leq)$ is an operation T satisfying $\forall x \in X$.*

1. $x \leq Tx$

2. $TTx = Tx$

3. $x \leq y$ *implies* $Tx \leq Ty$

An interior *operation has the same 2. and 3. but 1. is changed to*

1. $Tx \leq x$

Lemma 21 *In a partially ordered set (poset) $(X, \leq)$ a comonad is an interior operation.*

Lemma 22 $!A = A \wedge \mathbf{I}$

1. *is a comonad* $(!, e, id)$.
2. *such that there are isomorphisms* $!(A \times B) \cong !A \otimes !B$
3. *and* $!1 = \mathbf{I}$.

Proof: For 1. We must show that ! is an interior operator. In Lemma 12 we showed the first two requirements, here we show the third. Remember: On the the poset $H \times H^{op}$, $\leq$ is given by $\multimap$.

$$\begin{array}{lll}
(A \multimap B) & \text{implies} & !A \multimap !B \\
A \supset B \wedge \neg B \supset \neg A & \text{implies} & A \wedge \mathbf{I} \multimap B \wedge \mathbf{I} \\
A \supset B \wedge \neg B \supset \neg A & \text{implies} & A \wedge \mathbf{I} \multimap B \wedge \mathbf{I} \\
A \supset B \wedge \neg B \supset \neg A & \text{implies} & A \wedge \mathbf{I} \supset B \wedge \mathbf{I} \ \wedge \neg(B \wedge \mathbf{I}) \supset \neg(A \wedge \mathbf{I}) \\
A \supset B \wedge \neg B \supset \neg A & \text{implies} & A \supset B \wedge \mathbf{I} \ \wedge \neg B \vee \mathbf{I} \supset \neg A \vee \mathbf{I} \\
A \supset B \wedge \neg B \supset \neg A & \text{implies} & A \supset B \text{ and } A \supset \mathbf{I} \\
A \supset B \wedge \neg B \supset \neg A & \text{implies} & A \supset B
\end{array}$$

Now we use Lemma 21 to conclude ! gives rise to a comonad. It is immediately an endofunctor.

To show that there are isomorphisms $!(A \times B) \cong !A \otimes !B$ we need them to be equal in the algebra. However, $!(A \wedge B) = !A \otimes !B$ and $!1 = \mathbf{I}$ immediately by putting together the representation theorem and the projection theorem. ∎

Corollary 23 *The algebra of constructible duality is a model for classical linear logic.*

As we showed that N_c^- with our additional units has a representation onto $H \times H^{op}$. This means there is a sound translation from linear logic to constructible duality. Furthermore, this translation is the one alluded to: given by the table below.

Linear Logic	Constructible Duality
$\mathbf{0}$	$\mathbf{0}$
$\top$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{I}$
$\perp$	$\mathbf{I}$
$A \otimes B$	$\neg((\mathcal{A} \supset \neg\mathcal{B}) \wedge (\mathcal{B} \supset \neg\mathcal{A}))$
$A \multimap B$	$(\mathcal{A} \supset \mathcal{B}) \wedge (\neg\mathcal{B} \supset \neg\mathcal{A})$
$A \parr B$	$(\neg\mathcal{A} \supset \mathcal{B}) \wedge (\neg\mathcal{B} \supset \mathcal{A})$
$A \& B$	$\mathcal{A} \wedge \mathcal{B}$
$A \oplus B$	$\mathcal{A} \vee \mathcal{B}$
$A^{\perp}$	$\neg\mathcal{A}$
$!_l A$	$\mathcal{A} \wedge \mathbf{I}$
$?_l A$	$\mathcal{A} \vee \mathbf{I}$

where calligraphic letters indicate that the translation is recursively applied to the subformulas.

Therefore we have the simple theorem.

Theorem 24 *Let A be a formula in linear logic.*

If A is valid in linear logic, then $\mathcal{A}$ is valid in constructible duality where $\mathcal{A}$ is the translation from linear logic according to the table above.

This result can be strengthened for the multiplicatives.

Definition 25 *A formula is balanced if each propositional atom occurs at most twice and if twice with opposite sign.*

Every proof of a formula of multiplicative linear logic is an identification of a proof of a balanced formula.

Theorem 26 *If A is a balanced formula in multiplicative linear logic, then A is valid in multiplicative linear logic with mix, if and only if $\mathcal{A}$ is valid in constructible duality.*

The proof of this is in (Patterson and Costello 1997b)

In fact there is only one proof in each case, but that requires a proof of a coherence theorem for a suitable category but this is beyond the scope of this paper.

6 Conclusion

All of the logic transformers, !, ?, ¡, ¿ $!_l$, $?_l$ and $\sim\sim$ are closure or interior operators.

Closure and interior operations can be thought of as approximations. When the approximation is from above, like $\lceil n \rceil$, which maps every real number to the first natural greater than or equal to it, the approximation is a closure operation. $\lfloor n \rfloor$, the approximation from below it is an interior operation.

The Glivenko or Gödel $\sim\sim$ operator is a closure operation. Also Girard's *why not* $?_l$ operator is a closure operation as are our ? and ¿ operators.

The Gödel $\Box$ translation from S4 to intuitionistic logic is an interior operation. Girard's *of course* or $!_l$ and our ! and ¡ operators are also interior operations.

This shows that we can think logic transformers as approximations. Thus our projections (or exponentials) that recover consistent logics from paraconsistent logics are approximations by a consistent logic of a paraconsistent logic. Both ! and ¿ capture the notion of removing all negative information from a proposition and ? and ¡ capture the notion of removing all positive information from a proposition.

Thus we showed that we can naturally divide the exponential construction of Girard into two parts, one that deals with "linearity" or counting the occurrences of assumptions, and a second that captures the embedding of a intuitionistic logic in a paraconsistent logic.

We have four exponentials while linear logic has two and these occur in a conservative extension of N_c^- called constructible duality. They are all projections in the algebra, and the first two, the overdefined exponentials, arise as Girard's exponentials in a model of linear logic. The undefined exponentials do not appear in linear logic, but they are natural in constructible duality.

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