Souslin Absoluteness, Uniformization and Regularity properties of projective sets

Eyal Amir*
Dept. of Mathematics and Computer Science
Bar-Ilan University
52900 Ramat-Gan, Israel
amire@bimacs.cs.biu.ac.il

Haim Judah*
Dept. of Mathematics and Computer Science
Bar-Ilan University
52900 Ramat-Gan, Israel

February 18, 1995

Abstract

We show that Souslin Absoluteness and Projective Regularity holds iff Souslin Uniformization does. As a result, Souslin Absoluteness plus Σ^1_n Projective Regularity implies Δ^1_{n+1} Projective Regularity. Another result is that Σ^1_5 Souslin Absoluteness implies Δ^1_4 Projective Regularity, and Σ^1_6 Souslin Absoluteness implies Δ^1_5 Projective Regularity.

Contents

[1]We wish to thank the "Emmy Nother Institute in the Mathematics Department of Bar-Ilan University" for its financial support.

1 Introduction

We have several goals in our research in descriptive set theory. Two of them are:

1. To find a statement about the reals that explains completely the theory of the reals in Solovay models.
2. To find a combinatorial statement equivalent to "Projective measurability" (as well as the Baire Property).

In both of these goals, we introduce major advances. The following notion plays an important role (general references for this article might be [Je], [Je2], [Ku], [Mo], and [Ox]).

Notation 1.1 *In this article we deal only with the boldface classes in the projective hierarchy, so we will use the lightface symbols* (Σ^1_n) *for the boldface* ($\boldsymbol{\Sigma^1_n}$).

Definition 1.2 (cf **[JS]** §**0)** *We say that a forcing notion* $\mathbb{P}$ **is Souslin** *iff the set* $\mathbb{P}$ *and the relations* $\leq_{\mathbb{P}}$ *and* $\perp_{\mathbb{P}}$ *are all* Σ^1_1 *sets and* $\mathbb{P}$ *is ccc.*

As examples of Souslin Forcing notions, one has

$\mathbb{A}$ The Amoeba forcing notion.

$\mathbb{B}$ The Random forcing notion.

$\mathbb{C}$ The Cohen forcing notion.

$\mathbb{D}$ The Dominating forcing notion.

$\mathbb{E}$ The Amoeba meager (Universal Meager) forcing notion.

In the following, μ is the Lebesgue measure (on the appropriate field).

1.1 Latest Results

In a joint work of Judah and Bagaria ([BJ]), it was proved that in the Solovay model ([So]) Souslin Absoluteness holds.

Definition 1.3 (cf **[BJ], [Ju]** §**2)** *Let* V *be a universe of set theory. Given a forcing notion* $\mathbb{P} \in V$, *we say that* V *is* $\Sigma^1_n(\mathbb{P})$**-absolute** *iff for every* Σ^1_n*-sentence* φ *with parameters in* V *we have*

$$V \models \varphi \text{ iff } V^{\mathbb{P}} \models \varphi.$$

The corresponding definition for Π^1_n *is similar.*

We say that V *is* **Souslin Absolute**, *if it is* $\Sigma^1_n(\mathbb{P})$*-absolute for every Souslin Forcing notion* $\mathbb{P}$ *for every* $n \in \mathbb{N}$.

In the second direction of interest, we already have the following results:

Theorem 1.4 ([Ju2]) *1. Δ^1_2-measurability iff Σ^1_3 (Random)-Absolute*

2. Δ^1_2-categoricity iff Σ^1_3 (Cohen)-Absolute

3. Σ^1_2-measurability iff Σ^1_3 (Amoeba)-Absolute

4. Σ^1_2-categoricity iff Σ^1_3 (Hechler)-Absolute

Theorem 1.5 ([Ju2]) *1. $\Sigma^1_4(\mathbb{B})$-Absolute $+ \Sigma^1_3(\mathbb{A})$-Absolute $\rightarrow \Delta^1_3(L)$.*

2. $\Sigma^1_4(\mathbb{C})$-Absolute $+ \Sigma^1_3(\mathbb{D})$-Absolute $\rightarrow \Delta^1_3(B)$.

Shelah proved the following :

Theorem 1.6 (cf **[Ju2] p.8**) *$\Sigma^1_3(L) \Rightarrow (\forall r \in \mathbb{R})(\omega_1^{\mathbb{L}[r]} < \omega_1)$.*

Recently J. Brendle, using the ideas of [Ju2], proved the following

Theorem 1.7 (cf **[Ju2] p.14**) *Σ^1_4 (Amoeba)-Absolute $\rightarrow \Sigma^1_3$-measurability.*

Corollary 1.8 (cf **[Ju2] p.14**) *Σ^1_4 (Amoeba)-Absolute $\rightarrow \Sigma^1_3$-categoricity.*

Another recent result by L. Halbeisen and the second author in [HJ] is

Theorem 1.9 ([HJ]) *Σ^1_6-Mathias-absoluteness implies Σ^1_4-regularity for Ramsey ($\Sigma^1_4(\mathfrak{R})$).*

Looking at these results, the question arises whether there is a general connection between Souslin Absoluteness and Regularity (Measurability, Categoricity etc.). This will be the main motivation of the rest of this article.

2 Souslin Ideals - Foundations

Definition 2.1 *We say that a σ-ideal $\mathcal{I}$ is a* **Souslin Ideal** *if $\mathcal{I}$ is a nontrivial (i.e. $\mathbb{R} \notin \mathcal{I}$) Borel ccc absolute σ-ideal.*

Notice that this context is quite similar to the one used in [Ku2] with the difference that Kunen requires that his ideal (which he calls "reasonable ideal") will have a form of the Fubini property. We will abuse notations by referring to $\mathcal{I}$, which is defined on the **Borel** σ-algebra, as its expansion to the real line.

In [JuRo] H. Judah and A. Roslanowski tried to ensure the existence of "nice" Ideals, for some Forcing Notions.

Definition 2.2 *A forcing notion* $\mathbb{P}$ *is* **countably-1-generated** *if there are conditions* $p_n \in \mathbb{P}$ *(for* $n \in \omega$*) such that*

$$(\forall p \in \mathbb{P})(\forall q \in \mathbb{P}, q \perp p)(\exists n \in \omega)(p_n \perp p \ \&\ p_n \not\perp q).$$

In this situation the conditions p_n $(n \in \omega)$ *are called* σ*-1-generators of the forcing notion* $\mathbb{P}$*.*

They showed that for each forcing notion $\mathbb{P}$ which is countably-1-generated, there is a Souslin Ideal $\mathcal{I}_{\mathbb{P}}$ on ω^ω such that:

Corollary 2.3 *The quotient algebra* $\mathbf{Borel}(\omega^\omega)/\mathcal{I}_{\mathbb{P}}$ *is a ccc complete Boolean algebra and the mapping*

$$\mathbb{P} \longrightarrow \mathbf{Borel}(\omega^\omega)/\mathcal{I}_{\mathbb{P}} : p \mapsto [\phi(p)]_{\mathcal{I}_{\mathbb{P}}}$$

is a dense embedding (so $RO(\mathbb{P}) \cong \mathbf{Borel}(\omega^\omega)/\mathcal{I}_{\mathbb{P}}$*). For each Borel code c:* $[\![\dot{r} \in A_c]\!]_{\mathbb{P}} = [A_c]_{\mathcal{I}_{\mathbb{P}}}$ ■

For the rest of our work we will use the following context as our main point of view: Let $\mathbb{P} = \mathbf{Borel}(\omega^\omega)/\mathcal{I}$ be a Souslin forcing notion ($\mathcal{I}$ is a Souslin-ideal on ω^ω). All of the previous examples of Souslin Forcing notions are also examples of this situation. For Random and Cohen, the ideals are, correspondingly, the Null and the Meager sets.

Definition 2.4 *We say that a set* A *is* $\mathcal{I}$**-regular** *if there is a* **Borel** *set* B *such that* $B \triangle A \in \mathcal{I}$*. Let* $n \geq 1$*.* $\Sigma^1_n(\mathcal{I})$ *is the following statement:*
Every Σ^1_n *subset of the real line* A *is* $\mathcal{I}$*-regular.*

Our general $\mathbb{P}$ satisfies many nice properties. We give two examples.

Lemma 2.5 *Let* $\mathcal{M}$ *be a transitive model of ZFC. If* G *is an* $\mathcal{M}$*-generic filter on* $\mathbb{P}$*, then there is a unique real number* x_G *such that for all* $B \in \mathbb{P}$

$$x_G \in B^* \Leftrightarrow [B]_{\mathbb{P}} \in G \tag{1}$$

The formula (1) determines G *and hence* $\mathcal{M}[G] = \mathcal{M}[x_G]$*.*

Proof To start, we claim that there is at most one real number x that satisfies

$$\forall B \in \mathbf{Borel}(x \in B^* \Leftrightarrow [B] \in G). \tag{2}$$

If x satisfies (2), then x belongs to all B^* such that $[B] \in G$. If $x < y$ are two real numbers, let r be a rational number such that $x < r < y$, and let A be the interval $(r, \infty) \subseteq \mathbb{R}$. Either $[A]$ or $[\mathbb{R} \setminus A]$ belong to G, but $x \notin A^*$ and $y \notin (\mathbb{R} \setminus A)^*$.

In order to show that there exists a real number x that satisfies (2), let

$$x = sup\{r : r \text{ is a rational number and } [(r,\infty)] \in G\}. \tag{3}$$

By the genericity of G, there exists r such that $[(r,\infty)] \notin G$, and hence the supremum (3) exists. Note also that $x \notin \mathcal{M}$ (by the genericity of G). We shall show that x satisfies (2). We shall show, by induction on **Borel** codes in $\mathcal{M}$, that for every $c \in BC^{\mathcal{M}}$,

$$x \in A_c^* \iff [A_c] \in G. \tag{4}$$

First we consider the Σ_1^0-codes (in $\mathcal{M}$), and let us start with those $c \in \Sigma_1^0 \cap \mathcal{M}$ that code a rational interval, i.e., such that $c(n) = 1$ for exactly one n; then c codes the interval I_n. Let $I_n = (p,q)$. We have

$$\begin{aligned} x \in A_c^* &\iff p < x < q \\ &\iff p < sup\{r : [(r,\infty)] \in G\} < q \\ &\iff [(p,\infty)] \in G \wedge [(q,\infty)] \notin G \\ &\iff [(p,q)] \in G \iff [A_c] \in G. \end{aligned}$$

Now, if $c \in \Sigma_1^0$, then $A_c = \bigcup_{n=0}^{\infty} I_{k_n}$, where $\{k_n : n = 0,1,\ldots\}$ is the set $\{k : c(k) = 1\}$, and we have

$$\begin{aligned} x \in A_c^* &\iff x \in \bigcup_{n=0}^{\infty} I_{k_n}^* \\ &\iff \exists n (x \in I_{k_n}^*) \\ &\iff \exists n ([I_{k_n}] \in G) \\ &\iff \textstyle\sum_{n=0}^{\infty} [I_{k_n}] \in G \\ &\iff [\bigcup_{n=0}^{\infty} I_{k_n}] \in G \iff [A_c] \in G. \end{aligned}$$

Next let $\alpha < \omega_1^{\mathcal{M}}$, and let $c \in \Pi_\alpha^0 \cap \mathcal{M}$, and let us assume that (4) holds for all $c \in \Sigma_\alpha^0 \cap \mathcal{M}$. We may assume that $c(0) = 0$; then $u(c) \in \Sigma_\alpha^0 \cap \mathcal{M}$ and $A_{u(c)} = \mathbb{R} \setminus A_c$, and we have

$$x \in A_c^* \iff x \notin A_{u(c)}^* \iff [A_{u(c)}] \notin G \iff [A_c] \in G.$$

Finally, the induction step for Σ_α^0 is handled in a way similar to the case for $c \in \Sigma_1^0$. Thus (4) holds for every $c \in BC^{\mathcal{M}}$, and thus x is the unique real number that satisfies (1). ■

One should notice that in fact we did not use the absoluteness of the ideal, but just the structure of the partial order (being of the form **Borel**/$\mathcal{I}$). The following lemma provides a characterization of $\mathbb{P}$-reals.

Lemma 2.6 *A real number is a $\mathbb{P}$-real over $\mathcal{M}$ if and only if it does not belong to any Borel set $I \in \mathcal{I}$ with a code in $\mathcal{M}$.*

Proof On the one hand, if x is a $\mathbb{P}$-real over $\mathcal{M}$, let G be an $\mathcal{M}$-generic filter on $\mathbb{P}$ such that $x = x_G$. Then if $A_c \in \mathcal{I}$, then $[A_c] \notin G$, and by 2.5, $x \notin A_c^*$.

On the other hand, let x be such that $x \notin A_c^*$ whenever $A_c \in \mathcal{I}$ (and $c \in \mathcal{M}$). First we observe that if $[A_c] = [A_d]$ then $A_c \triangle A_d \in \mathcal{I}$, hence $A_c^* \triangle A_d^* \in \mathcal{I}^*$ (by the absoluteness of $\mathcal{I}$). It follows that x belongs to A_c^* if and only if x belongs to A_d^*. Let

$$G = \{[A_c] : x \in A_c^*\}. \tag{5}$$

It is easy to see that G is a filter on $\mathbb{P}$: If $[A_c] \in G$ and $[A_d] \in G$, then $x \in A_c^* \cap A_d^*$ and hence $[A_c \cap A_d] \in G$; similarly, if $[A_c] \geq [A_d]$ and $[A_c] \in G$, then $[A_d] \in G$ (recall that we use "$p \geq q$" to denote "p is stronger than q"). We shall show that G is $\mathcal{M}$-generic. Since $\mathbb{P}$ satisfies the ccc, it suffices to show that if $\{A_{c_n} : n \in \omega\} \in \mathcal{M}$ is such that $\sum_{n=0}^{\infty}[A_{c_n}] \in G$, then some $[A_{c_n}]$ is in G. But this is true because

$$\sum_{n=0}^{\infty}[A_{c_n}] = [\bigcup_{n=0}^{\infty} A_{c_n}] \text{ and } (\bigcup_{n=0}^{\infty} A_{c_n})^* = \bigcup_{n=0}^{\infty} A_{c_n}^*.$$

Finally, we claim that $x = x_G$. But this follows from (5), by the genericity of G. Thus a real number x is a $\mathbb{P}$-real over $\mathcal{M}$ if and only if $x \notin A_c^*$ for any **Borel** set $A_c \in \mathcal{I}^{\mathcal{M}}$. $\blacksquare$

Definition 2.7 *We say that σ is a* **$\mathbb{P}$-name for a $\mathbb{P}$-real** *over a model V, if G is a $\mathbb{P}$-generic filter over V and a is the intersection of G, and σ is the $\mathbb{P}$-name of a. a is called a* **$\mathbb{P}$-real**. *We will denote the set of all $\mathbb{P}$-reals over $\mathcal{M}$ by $Pr(\mathcal{M})$.*

Corollary 2.8

$$Pr(\mathcal{M}) = \mathbb{R}^* \setminus \bigcup\{A_c^* : c \in BC^{\mathcal{M}} \wedge A_c \in \mathcal{I}\}.$$

PROOF Notice that by the last lemma we get

$$Pr(\mathcal{M}) = \mathbb{R}^* \setminus \bigcup\{A_c^* : c \in BC^{\mathcal{M}} \wedge A_c^* \in \mathcal{I}\}.$$

Thus by the absoluteness of $\mathcal{I}$ we get

$$Pr(\mathcal{M}) = \mathbb{R}^* \setminus \bigcup\{A_c^* : c \in BC^{\mathcal{M}} \wedge A_c \in \mathcal{I}\}.$$

$\blacksquare$

In the rest of our work we will abuse notations and use the notations of subsets of the plane, also for their class in $\mathbb{P}$ (modulo the ideal $\mathcal{I}$). For example B will also denote $[B]_{\mathbb{P}}$.

3 Souslin Uniformization

3.1 General Facts

Lemma 3.1 *Let σ be a $\mathbb{P}$-name for a real number. Then there is a* **Borel** *function f such that for a $\mathbb{P}$-real a over V,*

$$V[a] \models \sigma[a] = f(a)$$

PROOF We define f by approximating it using simple functions. We work in $[0,1]$. Let $A_{i,n} = [\![\sigma \in (\frac{i}{2^n}, \frac{i+1}{2^n})]\!]$, $i < 2^n$. Let

$$f_n(x) = \sum_{i<2^n} \frac{i}{2^n} \chi_{A_{i,n}}(x)$$

where $\chi_{A_{i,n}}$ is the characteristic function on $A_{i,n}$. So, each f_n is a simple **Borel** function. Let

$$f(x) = \lim_{n\to\infty} f_n(x)$$

Since $f(x) = y \;\Leftrightarrow\; \forall n \exists m \forall k \geq m(\mid y - f_k(x) \mid < \frac{1}{n})$, f is **Borel**. Now, let a be a $\mathbb{P}$-real over V. Pick $\varepsilon > 0$. For every n, there is a unique $i < 2^n$ such that $a \in A_{i,n}$. But if $a \in A_{i,n}$, $\sigma[a] \in (\frac{i}{2^n}, \frac{i+1}{2^n})$. Also $f_n(a) = \frac{i}{2^n}$. Hence, $\mid \sigma[a] - f_n(a) \mid < \frac{1}{2^n}$. Thus, we can find n such that $\mid \sigma[a] - f_n(a) \mid < \varepsilon$. ∎

Lemma 3.2 *Let $n \geq 2$. Assume $\varphi(x)$ is a Π^1_n-formula and f is a* **Borel** *function (Graph(f) is* **Borel***). Then $\varphi(f(x))$ is also a Π^1_n-formula in the additional parameter, the Borel code of f.*

PROOF Saying that for x, $V \models \varphi(f(x))$ holds, is equivalent to saying

$$V \models (\forall x \exists y(\langle x, y\rangle \in Graph(f))) \wedge (\forall y(\langle x, y\rangle \in Graph(f) \Rightarrow \varphi(y))).$$

∎

3.2 Souslin Uniformization

We will show that there is a strong relationship between Uniformization and Souslin absoluteness.

Definition 3.3 *Let $n \geq 1$. $\Pi^1_n(\mathbb{P})$-uniformization (or as we will mention it* **Souslin Uniformization***) is the following statement:*
For every A a Π^1_n subset of the plane, if $\{x : A_x = \emptyset\} \in \mathcal{I}$, then there is a **Borel** *function $f : \mathbb{R} \to \mathbb{R}$ such that $\{x : f(x) \in A_x\}^c \in \mathcal{I}$.*
$\Sigma^1_n(\mathbb{P})$-uniformization is defined similarly. We will sometimes use $\Sigma^1_n(\mathcal{I})$ instead of $\Sigma^1_n(\mathbb{P})$ since this definition depends only on the ideal $\mathcal{I}$.

Corollary 3.4 *$\Sigma^1_n(\mathbb{P})$-uniformization iff $\Pi^1_{n-1}(\mathbb{P})$-uniformization.*

PROOF The forward direction is obvious. The backward direction is as follows: Take A a Σ^1_n subset of the plane, and assume that $\{x : A_x = \emptyset\} \in \mathcal{I}$. $A = \{(x,y) : \varphi(x,y)\}$ where φ is a Σ^1_n-formula. So $\varphi(x,y) = \exists z \psi(x,y,z)$, where ψ is a Π^1_{n-1}-formula. The idea is to use the function guaranteed from the $\Pi^1_{n-1}(\mathbb{P})$-uniformization to replace the "$\exists$" quantifier in φ. Take

$$\phi : \mathbb{R}^2 \xrightarrow{1:1 \text{ onto}} \mathbb{R}$$

to be a **Borel** function mapping $\mathbb{R}^2$ to $\mathbb{R}$. By our assumption on A, $E = \{x : A_x = \emptyset\} \in \mathcal{I}$. So,

$$(\forall x \notin E)(\exists y, z\ \psi(x,y,z)).$$

Therefore, we have the needed assumption for the set $B = \{\langle x, \phi(y,z)\rangle : \psi(x,y,z)\}$. But this is a Π^1_{n-1}-set, so by $\Pi^1_{n-1}(\mathbb{P})$-uniformization there is a function f such that

$$I = \{x : f(x) \notin B_x\} \in \mathcal{I} \tag{6}$$

Take $g(x,y) = x$. Then $F(x) = g(\phi^{-1}(f(x)))$ is the needed function $((\forall x \notin I)(F(x) \in A_x))$. ■

The following lemma is due to H. Woodin ([Wo] 1,2):

Lemma 3.5 (Woodin) *Let L, B denote the Null and Meager Ideals. Then*

1. *$\Pi^1_n(L)$-uniformization implies Σ^1_{n+2}-absoluteness for Random.*
2. *$\Pi^1_n(B)$-uniformization implies Σ^1_{n+2}-absoluteness for Cohen.*

We will now rephrase and prove that lemma for our more general case:

Lemma 3.6 *Let $n \geq 1$. $\Pi^1_n(\mathbb{P})$-uniformization implies Σ^1_{n+2}-absoluteness for $\mathbb{P}$.*

PROOF Let us prove the case $n = 1$. The general case follows by induction on the complexity of the formula.

Let $\exists x \forall y \varphi(x,y,z)$ be a Σ^1_3-formula with parameters in V, where φ is Σ^1_1. Suppose that v is a $\mathbb{P}$-real over V and $V[v] \models \exists x \forall y \varphi(x,y,a)$, for some $a \in \mathbb{R} \cap V$.

Let b be a witness so that $V[v] \models \forall y \varphi(b,y,a)$. Choose in V a term τ for b. τ may be chosen as a **Borel** function g such that

$$V[v] \models \forall y \varphi(g(v), y, a) \tag{7}$$

(see 3.1).

Suppose $V \models \forall x \exists y \neg\varphi(x,y,a)$. Then, $V \models \forall x \exists y \neg\varphi(g(x),y,a)$. Let $A = \{(x,y) : \neg\varphi(g(x),y,a)\}$. By $\Pi^1_1(\mathbb{P})$uniformization, there is a **Borel** function f such that $\{x : (g(x), f(x)) \in A\}^c \in \mathcal{I}$. Choose a **Borel** set $B \subseteq \{x : (g(x), f(x)) \in A\}$ such that $B^c \in \mathcal{I}$. Hence,

$$V \models \forall x (x \in B \Rightarrow \neg\varphi(g(x), f(x), a)).$$

Since $\neg\varphi$ is Π^1_1, $\forall x(x \in B \Rightarrow \neg\varphi(g(x), f(x), a))$ is Π^1_2 with the **Borel** codes for B, f, g as additional parameters (see 3.2). So,

$$V[v] \models \forall x(x \in B \Rightarrow \neg\varphi(g(x), f(x), a)).$$

But since v is a $\mathbb{P}$-real over V, and since the complement of B is a **Borel** set in the ideal $\mathcal{I}$ in V, $v \in B$ (see 2.6). Therefore, $V[v] \models \neg\varphi(g(v), f(v), a)$, which contradicts (7) above.

The other direction (in proving absoluteness) is simply by Shoenfields theorem (cf [Je] Ch. 41) which gives us Σ^1_2-absoluteness, and in the induction step - by the induction hypothesis. ■

Lemma 3.7 *Fix $n > 0$.*
Assume Π^1_{n+1}-absoluteness for $\mathbb{P}$. Take $\varphi \in (\Sigma^1_n \cup \Pi^1_n)$, and let τ be the canonical $\mathbb{P}$-name for a $\mathbb{P}$-real. Assign $p = \{x : \varphi(x)\}$. Then

1. *if p is $\mathcal{I}$-regular, then*
$$[\![\varphi(\tau)]\!] \;=\; p/\mathcal{I}.$$
2. *if p contains an $\mathcal{I}$-regular subset q, then*
$$[\![\varphi(\tau)]\!] \geq q/\mathcal{I}.$$

Proof Let us prove *2* first, and then *1* will follow easily. Take p and q as mentioned above. Take a **Borel** set $F \subseteq q$ such that $F/\mathcal{I} = q/\mathcal{I}$ (By the assumption on q, where $\mathcal{I}$ is the σ-ideal mentioned above). Take $r = F/\mathcal{I}$. We claim that $r \Vdash \varphi(\tau)$. Take $a \in F^*$ $\mathbb{P}$-real over V (if there is none, then $q \in \mathcal{I}$ and we are done). Then

$$V[a] \models \varphi(a),$$

since by $\Pi^1_{n+1}[\Pi^1_n]$-absoluteness for $\mathbb{P}$,

$$V \models \forall x \in F(\varphi(x)) \Rightarrow V^{\mathbb{P}} \models \forall x \in F(\varphi(x)).$$

But $V[a] \models \tau[a] = a$ (τ is the canonical $\mathbb{P}$-name for a $\mathbb{P}$-real). So $V[a] \models \varphi(\tau)$ (and this is for each $a \in F^*$ $\mathbb{P}$-real over V), thus $r \Vdash \varphi(\tau)$ and

$$[\![\varphi(\tau)]\!] \geq q/\mathcal{I}$$

■

Now, for the first part of the lemma, notice that p satisfies the assumptions given in the second part, of q. Thus it is obvious that

$$[\![\varphi(\tau)]\!] \geq p/\mathcal{I}$$

For equality, just observe that for '$\leq$' we have

$$[\![\neg\varphi(\tau)]\!] \geq \{x : \neg\varphi(x)\}/\mathcal{I}$$

(by the assumption of Π^1_{n+1}-absoluteness), which implies

$$[\![\varphi(\tau)]\!] \leq p/\mathcal{I}$$

Thus

$$[\![\varphi(\tau)]\!] = p/\mathcal{I}$$

■

Fact 3.8 *Let $A \subseteq \mathbb{R}$. $(A \cap [\![\varphi(\tau)]\!]) \notin \mathcal{I} \Rightarrow [\![\varphi(\tau)]\!] \notin \mathcal{I}$.*

PROOF $[\![\varphi(\tau)]\!] \in \mathcal{I} \Rightarrow A \cap [\![\varphi(\tau)]\!] \in \mathcal{I}$ ($\mathcal{I}$ is an ideal). ■

The following corollary is an application of the previous lemma.

Corollary 3.9 *Fix $n > 0$.*

1. *Assume $\Sigma^1_n(L)$. Assume also that Π^1_{n+1}-absoluteness for Random holds. Take $\varphi \in (\Sigma^1_n \cup \Pi^1_n)$ and let τ be the canonical random name for a random real. Then*
$$\mu([\![\varphi(\tau)]\!]) \;=\; \mu(\{x : \varphi(x)\})$$
2. *Assume $\Sigma^1_n(B)$. Assume further that Π^1_{n+1}-absoluteness for Cohen holds. Take $\varphi \in (\Sigma^1_n \cup \Pi^1_n)$ and let τ be the canonical Cohen name for a Cohen real. Then*
$$\{x : \varphi(x)\} \text{ is not meager} \iff [\![\varphi(\tau)]\!] \text{ is not meager}$$

■

We will now use the notion defined in 2.4 to prove a converse of 3.6.

Lemma 3.10 *$\Sigma^1_n(\mathcal{I})$ + $\Sigma^1_{n+2}(\mathbb{P})$-absoluteness implies $\Pi^1_n(\mathbb{P})$-uniformization.*

PROOF Let $A = \{(x, y) : \varphi(x, y)\}$ be a Π^1_n subset of the plane. Suppose that $\{x : A_x = \emptyset\} \in \mathcal{I}$. Let $C \in \mathcal{I}$ be a **Borel** set such that $\{x : A_x = \emptyset\} \subseteq C$. Let $B = \{(x, y) : x \in C\}$. Thus B is a **Borel** set in $\mathcal{I} \times \mathcal{P}(\mathbb{R})$. Let $\psi(x, y)$ be an arithmetical formula that defines B. Then

$$V \models \forall x(\exists y\varphi(x, y) \vee \exists y\psi(x, y))$$

By Σ^1_{n+2}-absoluteness for $\mathbb{P}$,

$$V^{\mathbb{P}} \models \forall x(\exists y\varphi(x, y) \vee \exists y\psi(x, y))$$

Let τ be the canonical name for a $\mathbb{P}$-real in V.

$$V^{\mathbb{P}} \models \exists y \varphi(\tau, y) \vee \exists y \psi(\tau, y)$$

Moreover, if a is a $\mathbb{P}$-real over V, then $V[a] \models \tau[a] = a$. But since $\{x : B_x \neq \emptyset\}$ is a **Borel** set contained in $\mathcal{I}$ in V, $a \notin \{x : B_x \neq \emptyset\}^*$. Hence

$$V^{\mathbb{P}} \models \exists y \varphi(\tau, y)$$

Let σ be a $\mathbb{P}$-name for a real such that

$$V^{\mathbb{P}} \models \varphi(\tau, \sigma)$$

Then we can find a Borel function f such that for each $\mathbb{P}$-real a, $V[a] \models \sigma[a] = f(a)$. So

$$V^{\mathbb{P}} \models \varphi(\tau, f(\tau)) \tag{8}$$

Now assume $\{x : \neg\varphi(x, f(x))\} \notin \mathcal{I}$. Take $p = [\![\neg\varphi(\tau, f(\tau))]\!]$. Take $a \in p^*$ a $\mathbb{P}$-real over V (one may do so, since by corollary 3.7, $p \notin \mathcal{I}$). Then, $V[a] \models \neg\varphi(a, f(a))$ (p forces that), but that contradicts (8) above.

Therefore $\{x : \neg\varphi(x, f(x))\} \in \mathcal{I}$. ■

Lemma 3.11 ([JuRo]) *Every analytic set is $\mathcal{I}$-regular.*

In order for our theorem to be complete, it remains to show that Uniformization implies Regularity.

Lemma 3.12 *$\Pi^1_n(\mathbb{P})$-uniformization implies $\Sigma^1_n(\mathcal{I})$.*

Proof Take a set $C \in \Sigma^1_n$. Take $A = (C^c \times \{0\}) \cup (C \times \{1\})$. By uniformization, we have a Borel function f that almost everywhere uniformizes A. So the preimage of 1 by f is **Borel** and is almost everywhere like C. ■

Theorem 3.13 *$\Sigma^1_{n+2}(\mathbb{P})$-absoluteness + $\Sigma^1_n(\mathcal{I})$ $\iff$ $\Pi^1_n(\mathbb{P})$-uniformization.*

Proof Obvious from lemmas 3.10, 3.6 and 3.12. ■

Corollary 3.14 *Using the previous theorem:*

1. *$\Sigma^1_{n+2}(\mathbb{B})$-absoluteness + $\Sigma^1_n(L)$ iff $\Pi^1_n(L)$-uniformization.*
2. *$\Sigma^1_{n+2}(\mathbb{C})$-absoluteness + $\Sigma^1_n(B)$ iff $\Pi^1_n(B)$-uniformization.*

■

Using these results we can now establish a new link between Souslin Absoluteness and Regularity.

Lemma 3.15 *Σ^1_n($\mathbb{P}$-uniformization) implies $\Delta^1_n(\mathcal{I})$.*

PROOF Take a set $C \in \Delta^1_n$. Take $A = (C^c \times \{0\}) \cup (C \times \{1\})$. A is a Σ^1_n-set (even Δ^1_n-set). By uniformization, we have a Borel function f that almost everywhere uniformizes A. So the preimage of 1 by f is **Borel** and is almost everywhere like C. ■

Corollary 3.16 *$\Sigma^1_{n+2}(\mathbb{P})$-absoluteness + $\Sigma^1_n(\mathcal{I}) \Rightarrow \Delta^1_{n+1}(\mathcal{I})$.*

PROOF By Theorem 3.13 we get $\Pi^1_n(\mathbb{P})$-uniformization. Using 3.15 and 3.4 we get $\Delta^1_{n+1}(\mathcal{I})$. ■

We tried to show the full Induction Step (i.e $\Sigma^1_n \Rightarrow \Sigma^1_{n+1}$) using Partial Functions Uniformization. But J. Brendle showed that there are cases where Uniformization for sets which are not "Full" in the meaning of the Souslin Ideal $\mathcal{I}$ (i.e the guarantee to find a function for each set A in the proper Step in the Projective Hierarchy) implies Σ^1_{n+1} Regularity, while the Uniformization itself is (in these cases) equivalent to Δ^1_n Regularity (take for instance the case of $n = 1$ and $\mathbb{P} = \mathbb{C}$). One might have suggested the use of Partial Functions, but we proved that the existence of a partial function (under an assumption of our basic Uniformization scheme) implies the existence of a Complete Function for that same set. Thus this way of inquiry will not bear more fruits.

Corollary 3.17 *Souslin-Absoluteness implies $\Delta^1_4(\mathbb{B})$ and $\Delta^1_4(\mathbb{C})$.*

PROOF Using 1.7 and 1.8 and then applying 3.16 leads to the corollary. ■

Problem 3.18 *It is still open whether Souslin Absoluteness implies $\Sigma^1_n(\mathcal{I})$.*

4 Acknowledgements

We would like to thank all of the people who read and verified the preprint version of this article including Saharon Shelah, Andrzej Roslanowski, Miroslav Repicky, Otmar Spinas, Tzvi Scarr, the referee and Tomek Bartoszyński. We would especially want to thank Joerg Brendle for his important corrections and remarks.

References

[BJ] J. Bagaria and H. Judah, **Amoeba forcing, Souslin forcing and additivity of measure**, *Set Theory of the continuum* (H. Judah, W. Just, H .Woodin, Eds.), Springer, 1992, pp 155-173.

[HJ] L. Halbeisen, H. Judah **Mathias Absoluteness and the Ramsey Property** submitted to the Journal of Symbolic Logic.

[Je] T. Jech **Set Theory** Academic Press, New York, 1978.

[Je2] T. Jech **Multiple Forcing** Cambridge Tracts in Mathematics No. 88, New York, 1985.

[Ju] H. Judah, **Souslin Absoluteness**, submitted to the proceedings of the conference in Honour of Pf. Azriel Levy IMCP.

[Ju2] H. Judah, **Absoluteness for projective sets**, *Logic Colloquium 1990* (J. Oikkonen, J. Väänänen, Eds.), Lecture Notes in Logic 2, Springer 1993, pp. 145-154.

[JuBa] H. Judah, T. Bartoszyński **Measure and Category in Set Theory — the Asymmetry**, Preprint.

[JuRo] H. Judah & A. Roslanowski **Ideals determined by some Souslin forcing notions**, preprint.

[JS] H. Judah, S. Shelah, **Souslin Forcing**, The Journal of Symbolic Logic, vol.53(1988):1182–1207.

[Ku] K. Kunen **Set Theory**, studies in Logic and the Foundations of Mathematics vol.102, North-Holland (1980).

[Ku2] K. Kunen. **Random and Cohen reals**, in: **Handbook of Set-Theoretic Topology** (K .Kunen, J.E. Vaughan eds), North-Holland 1984.

[Mo] Y.N.Moschovakis, **Descriptive Set Theory**, *Studies in logic and the foundations of mathematics*, vol. 100, North-Holland 1980.

[Ox] J.C.Oxtoby **Measure and Category**, Graduate texts in Mathematics, Springer-Verlag (1971).

[So] R.Solovay, **A model of set-theory in which every set of reals is Lebesgue measurable**, Ann.Math. 94 (1971), 201-245.

[Wo] W.H. Woodin **On the Consistency Strength of Projective Uniformization**, Proceedings of the Herbrand Symposium. Logic Colloquium '81. J.Stern (editor). 365-384. North-Holland Publishing Company, 1982.