Souslin Absoluteness, Uniformization and Regularity properties of projective sets

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Contents

Chapter 1

Introduction

In this introduction we give a brief historical account of the subjects and problems we shall be considering in the following chapters, as well as a summary of the main results.

1.1 Descriptive Set Theory

Descriptive Set Theory was developed by Lusin and others which continue Borel's and Lebesgue's work on real functions.

Borel sets were first introduced by **Borel** in 1905, which also proved that 1.1.3 below holds. A mistake that found Souslin in Borels work, lead to the construction of an Analytic nonborel set and the construction of the Souslin-operation $\mathcal{A}$ in 1917. The Projective sets were introduced by Lusin and sierpinski in 1925.

We will work in ZFC (Zermelo-Fraenkel Set Theory with the axiom of choice). Our basic references are [Je] and [Ku].

α, β, γ etc. denote ordinals (and sometimes cardinals). ω is the set of all finite, or natural numbers, which we denote by i, j, k, l, m, n. ω^ω is the **Baire Space**, i.e., the set of all infinite sequences of natural numbers with the product topology. 2^ω is the **Cantor Space** i.e., the set of all infinite sequences of zeroes and ones with the product topology. $\mathbb{R}$ is the real line with the topology generated by the open intervals with rational endpoints.

Real numbers are elements of $\mathbb{R}$, although by our previouse remarks, elements of 2^ω and ω will also be called real numbers or, for short, real. Usually f, g denote elements of ω^ω, or 2^ω.

The Spaces mentioned above are all polish spaces. i.e., separable complete metric spaces. If X is a polish space, then $A \subset X$ is a **Borel Set** if it belongs to the smallest σ-algebra of subsets of X containing all open sets. The following is a more explicit definition of Borel Sets:

Definition 1.1.1 *For each countable ordinal α, we define the collection Σ^0_α and Π^0_α of subsets of X:*

Σ^0_1 = *the collection of all open sets.*
Π^0_1 = *the collection of all closed sets.*
Σ^0_α = *the collection of sets $A = \cup_{n=0}^{\infty} A_n (\forall n \in \omega \exists \beta < \alpha (A_n \in \Pi^0_\beta))$.*
Π^0_α = *the collection of all complements of sets in Σ^0_α.*

Fact 1.1.2 *A is a* **Borel set** *if A in some Σ^0_α (or some Π^0_α).*

Lemma 1.1.3 $\forall \alpha > 0 \exists (U \subset (\omega^\omega)^2 \wedge (U \in \Sigma^0_\alpha) \wedge \forall A \in \Sigma^0_\alpha(\omega^\omega) \exists a \in \omega^\omega (A = U_a))$. *$U$ is called a Universal Set.*

The projective sets are defined as follows:

Definition 1.1.4 *For each $n \geq 1$, we define the collection $\Sigma^1_n, \Pi^1_n, \Delta^1_n$ of subsets of a polish space X as follows:*

Π^1_0 = *the collection of all closed sets.*
Σ^1_n = *the collection of all projections of all Π^1_{n-1}-sets in $X \times \omega^\omega$.*
Π^1_n = *the collection of all complements of sets in Σ^1_n.*
Δ^1_n = $\Sigma^1_n \cap \Pi^1_n$.

The sets belonging to one of the collections Σ^1_n, Π^1_n are called **Projective***. A set is* **Analytic** *if it belongs to Σ^1_1.*

Lemma 1.1.5 (cf **[Je] Ch.7**) *Let $n > 0$. Then*

$$(\exists U \subset (\omega^\omega)^2)(U \in \Sigma^1_n \wedge \forall A \in \Sigma^1_n(\omega^\omega) \exists a \in \omega^\omega (A = U_a)).$$

U is called a Universal Set.

We have several goals in our research in descriptive set theory. Two of them are:

1. To find a statement about the reals that explains completely the theory of the reals in Solovay models.

2. To find a combinatorial statement equivalent to "Projective measurability" (as well as the Baire Property).

Next we will describe our efforts in these directions.

1.1.1 Notion for forcing

A forcing notion $\mathbb{P} = \langle \mathbb{P}, \leq_\mathbb{P} \rangle$ is a set $\mathbb{P}$ together with a binary relation $\leq_\mathbb{P}$ on $\mathbb{P}$ which is reflexive and transitive. We do not explicitly require that $\mathbb{P}$ has a least element, although this is almost always the case. p, q denote elements of $\mathbb{P}$, or conditions. $\Vdash_\mathbb{P}$ denotes the forcing relation of $\mathbb{P}$. $p \leq_\mathbb{P} q$ means "q is stronger that p", or "q extends p". σ, τ denote $\mathbb{P}$-names. Standard names are denoted by $\hat{n}, \hat{a}, \hat{\mathbb{P}}$. If G is a $\mathbb{P}$-generic filter over some model V, then $\tau[G]$ denote the evaluations of τ by G in $V[G]$.

1.2 Ideals & Souslin Forcing

We are interested in representing a forcing notion as a quotient Boolean algebra $\mathbf{Borel}/\mathcal{I}$ for some ccc Borel σ-ideal $\mathcal{I}$. This is interesting because of several reasons. The first (and the main one) is that if we assume (or prove) some additional properties of the ideal $\mathcal{I}$ then we can use the well-developed machinery of $\mathcal{I}$-reals (cf [Ku2]). The second reason is that we have a nice description of reals, Borel sets etc in extensions via such algebras.

In the second Chapter we will introduce the reader to the basic definitions and facts about souslin forcing and forcing at large. We will also introduce the concept of a model being absolute under certain forcing extensions, and Souslin Absoluteness. We will give some examples, such as Random, Cohen and Amoeba forcing notions which will be very usefull in the rest of our work.

In the third Chapter we will make a survey of the main results in a late work by H. Judah and A. Roslanowski (cf [JuRo]) concerning that problem. By the end of this chapter we will also prove that the main theorems known to be true for Random and Cohen forcings will also be true for a generalized case.

In the rest of our work we will emphasize the generalization available for each of the theorems we will prove. This will be our main job in the Fourth chapter (the exapmles), and will also be shown in the 5th and 6th chapters.

1.3 Regularity Properties

Among the various regularity properties of sets of reals considered in the literature, the Lebesgue measurability and the Baire property have been the most thoroughly studied. They have been the object of continued mathematical interest for almost a hundred years. We will concentrate in these properties concerning projective sets.

We will use the following notions: we shall write $\Sigma^1_n(L)(\Pi^1_n(L), \Delta^1_n(L))$ if every $\Sigma^1_n(\Pi^1_n, \Delta^1_n)$ set of reals is Lebesgue measurable, and $\Sigma^1_n(B)(\Pi^1_n(B), \Delta^1_n(B))$ if every $\Sigma^1_n(\Pi^1_n, \Delta^1_n)$ set of reals has the property of Baire.

1.3.1 Examples

We want to have an intuition on the connections that can be established between Souslin absoluteness, Uniformization, and regularity properties for projective sets. We will do so by giving some examples, and investigate these examples through the above criterions. We will be moved by the need to generalize theorems which were proved for measure and category, to other ideals.

The first example will be the well known model of solovay. We will show that in the model of Solovay - Projective Regularity concerning souslin forcing notions of the form $\mathbf{Borel}/\mathcal{I}$ (where $\mathcal{I}$ is souslin), holds. The second will be the

Constructible universe $\mathbb{L}$. We will show that in $\mathbb{L}$, we have (for these forcing notions) a set of complexity Δ^1_2 which is not regular concerning $\mathcal{I}$.

1.3.2 Uniformization

In Chapter 5 we introduce the concept of a model having a uniformization property concerning an ideal $\mathcal{I}$. We already know (by a work of Woodin in [Wo]) that Random-Uniformization implies Random-Absoluteness and Cohen-Uniformization implies Cohen-Absoluteness. We will generalize this fact to forcing notions of the form **Borel**/$\mathcal{I}$ having some properties, and prove that $\mathbb{P}$-Uniformization holds if and only if $\mathbb{P}$-Absoluteness and $\mathcal{I}$-regularity holds (i.e - the reverse direction to Woodin).

1.3.3 Regularity

It follows from Shoenfield's Absoluteness theorem that every model of ZF+DC is Σ^1_2-absolute under all forcing extensions. However, we can get more for special Souslin forcing notions.

Theorem 1.3.1 ([Ju2]) *1. Δ^1_2-measurability iff Σ^1_3 (Random)-Absolute.*

2. Δ^1_2-categoricity iff Σ^1_3 (Cohen)-Absolute.

3. Σ^1_2-measurability iff Σ^1_3 (Amoeba)-Absolute.

4. Σ^1_2-categoricity iff Σ^1_3 (Hechler)-Absolute.

Theorem 1.3.2 ([Ju2]) *1. Σ^1_4(Random)-Absolute + Σ^1_3(Amoeba)-Absolute $\rightarrow \Delta^1_3$-measurability*

2. Σ^1_4 (Cohen)-Absolute + Σ^1_3 (Hechler)-Absolute $\rightarrow \Delta^1_3$-categoricity

Shelah proved the following :

Theorem 1.3.3 (cf **[Ju2] p.8**) $\Sigma^1_3(L) \Rightarrow (\forall r \in \mathbb{R})(\omega_1^{\mathbb{L}[r]} < \omega_1)$.

Recently J. Brendle, using the ideas of [Ju2], proved the following

Theorem 1.3.4 (cf **[Ju2] p.14**) *Σ^1_4 (Amoeba)-Absolute $\Rightarrow$ Σ^1_3-measurability holds.*

Corollary 1.3.5 (cf **[Ju2] p.14**) *Σ^1_4 (Amoeba)-Absolute $\Rightarrow \Sigma^1_3$-categoricity.*

In our last chapter - we will establish a connection between Souslin absoluteness and Projective measurability/categoricity. We will show (using the results about uniformization) that Souslin absoluteness implies $\Delta^1_4(L)$ and $\Delta^1_4(B)$. We will also establish the general case.

Chapter 2

Souslin Forcing

In the following chapter we will introduce the reader to the common definitions and facts we will use in the rest of our work. We will, though, assume that the reader is well known with the methods of Forcing, and Descriptive Set Theory. Furthermore, notice that some of the references are to the original papers. In some cases these references are not clear to understand, and in others the notation used today is not the notation used in the original article (This turns mostly towards [MS] in which most of our well known partial orders Random, Cohen etc. and MA (Martin's Axiom) are not referred by their modern names, and even not properly defined). We will use in some cases a modern resource for the reader's comfort.

Notation: For general set theory, we refer the reader to [Je] and [Ku]. We will use $\mathbb{P}$ and $\mathbb{Q}$ as partial orders (we confuse to a certain extent Boolean-valued models $V^{\mathbb{P}}$ and generic extension $V[G]$, where G is $\mathbb{P}$-generic over V). For a formula φ of the forcing language for some p.o. $\mathbb{P}$, $[\![\varphi]\!]$ denotes the truth value of φ (in p.o. $\mathbb{P}$); furthermore, we sometimes use $\hat{n}, \hat{m}$, etc., to denote the canonical $\mathbb{P}$-names for the natural numbers n, m. For a function $i : A \to B$ we use i'' as the relative function from $\mathcal{P}(A)$ to $\mathcal{P}(B)$, where $i(X) = \{i(x) : x \in X\}$ for $X \subseteq A$. At the end of the chapter we will give some more notations, which will only then be valid.

2.1 Forcing

The two fundamental theorems of the method of forcing, the forcing theorem and the generic model theorem, are due to Cohen (To be accurate, Cohen's original method was formulated for particular examples of a notion of forcing, and under the assumption that $\mathcal{M}$ is a countable transitive model of ZFC). The boolean-valued version of Cohen's method has been formulated by Scott, Solovay, and Vopěnka. Following an observation of Solovay that the forcing relation can be viewed as assigning Boolean values to formulas, Scott formulated his version of Boolean-valued models in [Sc]. Vopěnka developed a theory of Choen's method of forcing, using open sets in a topological space as forcing conditions (in [Vo1], [Vo2], [Vo3], and [VuHa]), eventually arriving at the Boolean-valued version of forcing more or less identical to Scott-Solovay's version ([Vo4]).

Definition 2.1.1 (cf **[Ku] Ch.7 §7 p.218)** *Let* $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, \infty_q \rangle$ *and* $\langle \mathbb{P}, \leq_{\mathbb{P}}, \infty_p \rangle$ *be p.o's, and* $i : \mathbb{Q} \to \mathbb{P}$. i *is a* **complete embedding** *iff*

1. $\forall q, q' \in \mathbb{Q}(q' \leq q \Rightarrow i(q') \leq i(q))$.
2. $\forall q, q' \in \mathbb{Q}(q' \perp q \Leftrightarrow i(q') \perp i(q))$.
3. $\forall p \in \mathbb{P} \exists q \in \mathbb{Q} \forall q' \in \mathbb{Q}(q' \geq q \Rightarrow i(q') \not\perp p)$.

In (3) we call q *a* **reduction of** p **to** $\mathbb{Q}$.

Definition 2.1.2 (cf **[Je2] p.3)** *A set* D *is* **predense** *if*

$$(\forall p \in \mathbb{P})(\exists d \in D)(\exists q \in \mathbb{P})(q \geq d \wedge q \geq p).$$

Corollary 2.1.3 *In the last definition,* $i''(\mathbb{Q})$ *is predense in* $\mathbb{P}$. ■

Definition 2.1.4 (cf **[Ku] Ch.7 §7 p.218)**
$\langle \mathbb{Q}, \leq_{\mathbb{Q}}, \infty_q \rangle \subset_c \langle \mathbb{P}, \leq_{\mathbb{P}}, \infty_p \rangle$, *or* $\mathbb{Q} \subset_c \mathbb{P}$, *or* $\mathbb{Q} <_c \mathbb{P}$ *or* $\mathbb{Q} \lessdot \mathbb{P}$ *iff* $\leq_{\mathbb{Q}} = (\leq_{\mathbb{P}} \cap \mathbb{Q} \times \mathbb{Q})$, *and the inclusion (identity) map from* $\mathbb{Q}$ *to* $\mathbb{P}$ *is a complete embedding.*

Fact 2.1.5 (cf **[Ba] p.5-6)** *In the definition of complete embedding above, assume that i is 1:1. Then one may replace the second and third requirements of the definition with the requirement that i preserves maximal antichains. (i.e., for every maximal antichain* A *of* $\langle \mathbb{Q}, \leq_{\mathbb{Q}} \rangle$, *the corresponding set* $\{i(x) : x \in A\}$ *is a maximal antichain of* $\rangle \mathbb{Q}, \leq_{\mathbb{Q}} \langle$*).*

Theorem 2.1.6 (cf **[Ku] Ch.7 §7 p.220)** *Suppose* $i, \mathbb{P}, \mathbb{Q}$ *are in* $\mathcal{M}$, $i : \mathbb{P} \to \mathbb{Q}$, *and* i *is a complete embedding. Let* H *be* $\mathbb{Q}$*-generic over* $\mathcal{M}$. *Then* $i^{-1}(H)$ *is* $\mathbb{P}$*-generic over* $\mathcal{M}$ *and* $\mathcal{M}[i^{-1}(H)] \subseteq \mathcal{M}[H]$.

Definition 2.1.7 (cf **[Ku] Ch.7 §7 p.221)** *Let* $\mathbb{P}, \mathbb{Q}$ *be partial orders and* $i : \mathbb{P} \to \mathbb{Q}$. i *is a* **dense embedding** *iff*

1. $\forall p, p' \in \mathbb{P}(p' \leq p \Rightarrow i(p') \leq i(p))$.
2. $\forall p, p' \in \mathbb{P}(p' \perp p \Rightarrow i(p') \perp i(p))$.
3. $i''\mathbb{P}$ *is dense in* $\mathbb{Q}$.

Corollary 2.1.8 *Every dense embedding is a complete embedding.*

PROOF If $q \in \mathbb{Q}$, and $p \in \mathbb{P}$ with $i(p) \geq q$ is a reduction of q to $\mathbb{P}$. ■

Theorem 2.1.9 (cf **[Ku] Ch.7 §7 p.221**) *Suppose $i, \mathbb{P}, \mathbb{Q}$ are in $\mathcal{M}$, $i : \mathbb{P} \to \mathbb{Q}$, and i is a dense embedding. If $G \subseteq \mathbb{P}$, let*

$$j(G) = \{q \in \mathbb{Q} : \exists p \in G(i(p) \geq q)\}.$$

Then

1. *If G is $\mathbb{P}$-generic over $\mathcal{M}$, then $j(G)$ is $\mathbb{Q}$-generic over $\mathcal{M}$ and $G = i^{-1}(j(G))$.*
2. *If H is $\mathbb{Q}$-generic over $\mathcal{M}$, then $i^{-1}(H)$ is $\mathbb{P}$-generic over $\mathcal{M}$ and $H = j(i^{-1}(G))$.*
3. *In 1 or 2, if $G = i^{-1}(H)$ (or equivalently, if $H = j(G)$), then*

$$\mathcal{M}[G] = \mathcal{M}[H].$$

Definition 2.1.10 *We say that σ is a* **$\mathbb{P}$-name for a $\mathbb{P}$-real** *over a model V, if G is a $\mathbb{P}$-generic filter over V and a is the intersection of G, and σ is the $\mathbb{P}$-name of a. a is called a* **$\mathbb{P}$-real**. *We will denote the set of all $\mathbb{P}$-reals over $\mathcal{M}$ by $Pr(\mathcal{M})$.*

Definition 2.1.11 *We say that τ is the* **canonical $\mathbb{P}$-name for a $\mathbb{P}$-real** *over V, if τ is a $\mathbb{P}$-name such that for each a a $\mathbb{P}$-real over V, $V[a] \models \tau[a] = a$.*

Corollary 2.1.12 (cf **[Ku] Ch.7 §2**) . *For each such $\mathbb{P}$, there is a canonical $\mathbb{P}$-name for a $\mathbb{P}$-real.*

Definition 2.1.13 (cf **[JS] §0**) *Let $\mathbb{P}$ be a forcing notion. We say that $\tau \in V^{\mathbb{P}}$ is a* **simple $\mathbb{P}$-name** *for a real iff*

1. *the elements of τ are of the form $\langle p, \hat{n}, \hat{m}\rangle$ where $p \in \mathbb{P}$ and $\hat{n}, \hat{m}$ are the canonical $\mathbb{P}$-names for the natural numbers n, m. (this means $p \Vdash \tau(\hat{n}) = \hat{m}$)*
2. *for every $n \in \omega$, the set $\{p \in \mathbb{P} : \exists m \in \omega(\langle p, \hat{n}, \hat{m}\rangle \in \tau)\}$ is a maximal antichain of $\mathbb{P}$.*

Fact 2.1.14 *For every p.o. $\mathbb{P}$, and for every $\mathbb{P}$-name τ for a real (i.e., $\Vdash_{\mathbb{P}}$ "τ is a real"), there is a simple $\mathbb{P}$-name for a real σ such that $\Vdash_{\mathbb{P}}$ "$\tau = \sigma$".*

PROOF Fix $\mathbb{P}, \tau$, and suppose $\Vdash_{\mathbb{P}}$"τ is a real". For every $n \in \omega$, let A_n be a maximal antichain of $\mathbb{P}$ such that for every $p \in A_n$, $p \Vdash$"$\tau(\hat{n}) = \hat{m}$", for some $m \in \omega$. A_n exists since $\Vdash_{\mathbb{P}}$"τ is real" and, therefore, for every condition $p \in \mathbb{P}$ there is $q \geq p$ and $m \in \omega$ such that $q \Vdash$"$\tau(\hat{n}) = \hat{m}$". Now, let σ be the simple name defined by $\langle p, \hat{n}, \hat{m}\rangle \in \sigma$ iff $p \in A_n$ and $p \Vdash_{\mathbb{P}}$"$\tau(\hat{n}) = \hat{m}$". ■

Lemma 2.1.15 (cf **[Ku] Ch.7 §6 or [Ba] p.6)**) *Let $\mathbb{P}$ and $\mathbb{Q}$ be partial orders such that $\mathbb{P} \lessdot \mathbb{Q}$, and let τ be a simple $\mathbb{P}$-name for a real. Then, (we identify the standard $\mathbb{P}$-names for natural numbers with the corresponding standard $\mathbb{Q}$-names) τ is a $\mathbb{Q}$-name for a real. Moreover, if $G \subseteq \mathbb{Q}$ is a generic filter, then $\mathbb{P} \cap G$ is a generic filter for $\mathbb{P}$, and $\tau[\mathbb{P} \cap G] = \tau[G]$*

Definition 2.1.16 (cf **[Ba] p.19)**) *A partial order $\mathbb{P}$ is* **σ-centered** *iff there exists $h : \mathbb{P} \to \omega$ such that*

$$\forall n \in \omega \forall F \in [\mathbb{P}]^{<\omega}(\forall p \in F(h(p) = n) \Rightarrow \exists q \in \mathbb{P} \forall p \in F(q \geq p)).$$

We call the partition induced by h on $\mathbb{P}$ a σ-centering partition of $\mathbb{P}$.

2.2 Souslin Forcing

Souslin forcing, i.e., forcing notions in which the set of conditions, the ordering, and the incompatibility relation, are κ-Souslin sets of reals in the sense of Descriptive Set Theory, was first studied by H.Judah and S.Shelah in [JS]. In their paper, they show that Souslin forcing notions admit a systematic treatment, and specially so for $\aleph_0$-Souslin, i.e., the set of conditions is a Σ^1_1-set of reals, and the ordering and incompatibility relations are Σ^1_1. In this work we shall be interested only in $\aleph_0$-Souslin forcing notions which satisfy the countable antichain condition (ccc). Henceforth, for simplicity of notations, "Souslin" will mean $\aleph_0$-Souslin.

We will begin by giving some definitions and basic facts about Souslin forcing which we will use later in the rest of our work.

Definition 2.2.1 (cf **[BJ], [Ju] §2)**) *Let V be a universe of set theory. Given a forcing notion $\mathbb{P} \in V$, we say that V is* **$\Sigma^1_n(\mathbb{P})$-absolute** *iff for every Σ^1_n-sentence φ with parameters in V we have*

$$V \models \varphi \text{ iff } V^{\mathbb{P}} \models \varphi.$$

We are interested in this notion only in case of ccc forcing notions having an easy definition.

Definition 2.2.2 (cf **[JS] §0)**) *We say that a forcing notion $\mathbb{P}$* **has a Souslin-definition** *iff there are Σ^1_1-relations $R_0 \subseteq \omega^\omega$, and $R_1, R_2 \subseteq \omega^\omega \times \omega^\omega$, such that*

$$\begin{aligned} \mathbb{P} &= R_0 \\ \leq_{\mathbb{P}} &= \{\langle p, q\rangle; p \in \mathbb{P} \wedge q \in \mathbb{P} \wedge p \leq q\} = R_1 \\ \perp_{\mathbb{P}} &= \{\langle p, q\rangle; p \in \mathbb{P} \wedge q \in \mathbb{P} \wedge \forall r(r \not\leq p \vee r \not\leq q)\} = R_2. \end{aligned}$$

We say that $\mathbb{P}$ is **Souslin** *iff $\mathbb{P}$ is ccc and has a Souslin definition.*

Corollary 2.2.3 *Let $\mathbb{P}$ be Souslin, then $\perp_{\mathbb{P}}$ is in fact Borel for any Souslin forcing.*

PROOF $x \perp_{\mathbb{P}} y$ iff $\exists z(x \leq_{\mathbb{P}} z \wedge y \leq_{\mathbb{P}} z)$. Hence, both $\not\perp_{\mathbb{P}}$ and its complement $\perp_{\mathbb{P}}$ are Σ^1_1 subsets of $\omega^\omega \times \omega^\omega$, thus they are both Borel by the Souslin theorem (cf [Je], Theorem 93, Ch.39 p.502).

Fact 2.2.4 *Let $\mathbb{P}$ be a Souslin p.o and let $\mathcal{M}$ be a transitive model of ZF which contains the parameters of the Σ^1_1 formulas that define $\mathbb{P}$. Then, $\mathbb{P}$ is absolute for $\mathcal{M}$, i.e., $\mathbb{P}^{\mathcal{M}} = \mathbb{P} \cap \mathcal{M}$, $\leq^{\mathcal{M}}_{\mathbb{P}} = \leq_{\mathbb{P}} \cap \mathcal{M}$ and $\perp^{\mathcal{M}}_{\mathbb{P}} = \perp_{\mathbb{P}} \cap \mathcal{M}$.*

PROOF Follows obviously from the absoluteness of Σ^1_1-formulas for such models of ZF (cf [Je], Ch.40 p.509). ■

Lemma 2.2.5 *If $\mathbb{P}$ is a Souslin ccc p.o., then every antichain of $\mathbb{P}$ can be coded by a real, and the predicate, "x codes a maximal antichain of $\mathbb{P}$" is Π^1_1.*

PROOF Since $\mathbb{P}$ is ccc, every antichain A is a countable set of reals. So, we can write A as a sequence $\langle p_n : n < \omega \rangle$. But any such a sequence of reals can be recursively coded by a real. e.g., let $J : \omega \times \omega \to \omega$ be standard, 1:1 and onto, pairing function given by $J(n,m) = 2^n(2m+1) - 1$. Now, take $a \in \omega^\omega$ which satisfies $a(J(n,m)) = p_n(m)$. This is the desired code.

Now, for the other part, "x codes a maximal antichain of $\mathbb{P}$" iff

1. antichain: $\forall n, m(\{\langle i,j \rangle : x(J(n,i)) = j\} \perp_{\mathbb{P}} \{\langle i,j \rangle : x(J(n,i)) = j\})$.
2. maximal: $\neg \exists y \in \omega^\omega \forall n (y \perp_{\mathbb{P}} \{\langle i,j \rangle : x(J(n,i)) = j\})$.

Since $\mathbb{P}$ is Souslin, $\perp_{\mathbb{P}}$ is Borel (see 2.2.3). So 1 is Δ^1_1, and 2 is Π^1_1. ■

Definition 2.2.6 (cf **[BJ]**) *We say that V is* **Souslin-absolute** *if and only if V is $\Sigma^1_n(\mathbb{P})$-absolute for all $n \in \omega$ and all Souslin forcing notions $\mathbb{P}$.*

In [Ju2] §1 Haim Judah showed that Souslin-absoluteness is preserved by Souslin forcing; i.e., if V is Souslin-absolute and $\mathbb{P}$ is a Souslin forcing notion, then $V^{\mathbb{P}}$ is Souslin-absolute as well. One of the main technical devices for that proof (and many other) is the following result of [JS].

Theorem 2.2.7 ([JS], 3.14) *Let $\mathbb{P}$ be a forcing notion having a Souslin definition. Then $\mathbb{P}$ is ccc iff for every transitive model $\mathcal{M}$ containing $\mathbb{P}$, $\mathcal{M} \models$ "$\mathbb{P}$ is ccc" iff for some transitive model M containing $\mathbb{P}$, $M \models$ "$\mathbb{P}$ is ccc".*

Corollary 2.2.8 *Let $\mathbb{P}$ be a Souslin forcing notion. Then, the following are equivalent:*

1. *$\mathbb{P}$ is ccc.*
2. *For every transitive model $\mathcal{M}$ of ZF (including WF) with $\mathbb{P} \in \mathcal{M}$, $\mathcal{M} \models$ "$\mathbb{P}$ is ccc".*

3. *There exists a transitive model of ZF (including WF) with $\mathbb{P} \in \mathcal{M}$ such that $\mathcal{M} \models$ "$\mathbb{P}$ is ccc".*

PROOF It is clear that *2* implies *3* (assuming consistency of ZF of course). *1* implies *2*, and *3* implies *1*, follow immediately from the theorem. ■

In [BJ] it was proved that if we collapse an inaccessible cardinal to ω_1, and we build the Solovay Model, then this model satisfies "Souslin Absoluteness". Thus the following result was obtained.

Theorem 2.2.9 ([BJ]) *The following theories are equiconsistent:*

1. *ZFC + there is an inaccessible cardinal*
2. *ZFC + Souslin-Absolute*
3. *ZFC + Σ^1_3-measurability*

Lemma 2.2.10 *For a forcing $\mathbb{P}$ having a Souslin definition, the statement "$\mathbb{P}$ is ccc" is a Δ^1_2-statement.*

PROOF By theorem 2.2.8, "$\mathbb{P}$ is ccc" iff

$\forall\mathcal{M}$ (($\mathcal{M}$ is a countable transitive model of $ZFC^* \ \wedge\ \mathbb{P} \in \mathcal{M}$) $\rightarrow\ \mathcal{M} \models$ "$\mathbb{P}$ is ccc") iff

$\exists\mathcal{M}$ ($\mathcal{M}$ is a countable transitive model of $ZFC^* \ \wedge\ \mathbb{P} \in \mathcal{M}$ $\wedge\ \mathcal{M} \models$ "$\mathbb{P}$ is ccc").

The first formula is Π^1_2, while the second one is Σ^1_2. ■

Lemma 2.2.11 *The statement "b encodes a Souslin forcing" is a Π^1_2-statement.*

PROOF Let $\varphi(x, y, z)$ be a universal Σ^1_1-formula, then for every Σ^1_1-formula $\Psi(x, y)$ there is an $a \in \omega^\omega$ such that

$$\varphi(x, y, a) \ \Leftrightarrow\ \Psi(x, y).$$

Furthermore, given $b \in \omega^\omega$, set

$$b_i(n) := b(3n + i)(i \in \{0, 1, 2\}).$$

Given $b \in \omega^\omega$, we define the triple $(\mathbb{P}_b, \leq_{\mathbb{P}_b}, \perp_{\mathbb{P}_b}) = (R_0, R_1, R_2)$ as follows:

1. $R_0 = \mathbb{P}_b = \{x : \exists y(\varphi(x, y, b_0))\}$.
2. $R_1 = \leq_{\mathbb{P}_b} = \{\langle x, y\rangle : \varphi(x, y, b_1)\}$.
3. $R_2 = \perp_{\mathbb{P}_b} = \{\langle x, y\rangle : \varphi(x, y, b_2)\}$.

Thus, "b encodes a Souslin forcing" iff "$(\mathbb{P}_b, \leq_{\mathbb{P}_b}, \perp_{\mathbb{P}_b})$ is a Souslin forcing" iff the following are satisfied:

1. $\langle R_0, R_1\rangle$ is a partial order:
$$\begin{aligned}\forall x, y, z \quad & ((R_0(x) \wedge R_0(y) \wedge R_0(z)) \Rightarrow \\ & [R_1(x,x) \wedge (((R_1(x,y) \wedge R_1(y,z)) \Rightarrow R_1(x,z)) \\ & \wedge ((R_1(x,y) \wedge R_1(y,x)) \Rightarrow x = y)])).\end{aligned}$$
2. R_2 is incompatibility:
$$\begin{aligned}& \forall x, y((R_0(x) \wedge R_0(y)) \Rightarrow [R_2(x,y) \\ \Leftrightarrow\ & \forall z(R_0(z) \rightarrow (\neg R_1(z,x) \vee \neg R_1(z,y)))]).\end{aligned}$$
3. Field $(\leq_{\mathbb{P}_b}) \subseteq \mathbb{P}_b$, field $(\perp_{\mathbb{P}_b}) \subseteq \mathbb{P}_b$:
$$\forall x(\exists y(R_1(x,y) \vee R_2(x,y) \vee R_1(y,x) \vee R_2(y,x)) \ \Rightarrow \ R_0(x))$$
4. $\mathbb{P}_b$ satisfies ccc

1 through *3* are easily seen to be Π^1_2; *4* is Δ^1_2 by the preceding Lemma (2.2.10). ■

2.3 Examples for Souslin forcing notions

Let us consider generic extensions of a ground model $\mathbf{V}$, using either the algebra of **Borel** sets modulo the ideal of null sets or the algebra of **Borel** sets modulo the ideal of meager sets. The reader may take a look in [MS] (p.153, pp.166-167), [BJS], [JRS], [BJ] and [JuBa] Ch.3 (besides the references below).

Definition 2.3.1 ([So]) *Let $\mathcal{N}$ be the σ-ideal of null sets*

$$\mathcal{N} = \{B \in \mathbf{Borel} : \mu(B) = 0\}$$

Random forcing $\mathbb{B} = \mathbf{Borel}/\mathcal{N}$, *is the Boolean algebra of all the* **Borel** *subsets of the unit interval* $[0,1]$ *modulo the ideal of the* **Borel** *sets of Lebesgue measure zero (the Null sets). The ordering is given by $[A] \leq [B]$ iff $\mu(B \setminus A) = 0$, where μ is the Lebesgue measure.*

Fact 2.3.2 (cf **[Je] Ch.42 p.542-544)** *If G is a generic filter for the Random forcing over some model V, then The intersection of the filter $\bigcap\{A^{V[G]} : [A] \in G\}$ contains a single real called a Random real over V. Moreover, if $\{r\} = \bigcap\{A^{V[G]} : [A] \in G\}$ is a Random real over V, then for every $[A] \in \mathbb{B}^V$,*

$$V[G] \models \text{“}[A] \in G \Longleftrightarrow r \in A^{V[G]}\text{”}.$$

Hence $V[G] = V[r]$.

Fact 2.3.3 (cf **[Je] Ch.42 p.544)** *A real is a Random real over a model V iff it does not belong to any* **Borel** *set of measure zero with code in V.*

Fact 2.3.4 (cf **[Ba] p.15-16)** *The Random forcing notion is Souslin.*

Definition 2.3.5 (cf **[Je] Ch.42 p.542-544)** *Let $\mathcal{M}$ be the σ-ideal of Meager sets (sets of first category)*

$$\mathcal{M} = \{B \in \mathbf{Borel} : B \textit{ is meager}\}$$

Cohen forcing $\mathbb{C} = \mathbf{Borel}/\mathcal{M}$, *is the Boolean algebra of all the* **Borel** *subsets of the unit interval* $[0,1]$ *modulo the ideal of the* **Borel** *sets of first category. (the Meager sets). The ordering is given by* $[A] \leq [B]$ *iff* $B \setminus A$ *is of first category.*

The original definition of the Cohen forcing notion (which is isomorphic to our modern one (see [Ba] p.14)) is as follows:

Definition 2.3.6 ([Co]) *The conditions are finite sequences of zeroes and ones ordered by inclusion.*

We will use the modern one.

Fact 2.3.7 (cf **[Je] Ch.42 p.542-544)** *If G is a generic filter for the Cohen forcing over some model V, then the intersection of the filter $\bigcap\{A^{V[G]} : [A] \in G\}$ contains a single real called a* **Cohen real** *over V. Moreover, if $\{c\} = \bigcap\{A^{V[G]} : [A] \in G\}$ is a Cohen real over V, then for every $[A] \in \mathbb{C}^V$,*

$$V[G] \models \text{“}[A] \in G \iff c \in A^{V[G]}\text{”}.$$

Hence $V[G] = V[c]$.

Fact 2.3.8 (cf **[Je] Ch.42 p.544)** *A real is a Cohen real over a model V iff it does not belong to any meager* **Borel** *set with code in V.*

Fact 2.3.9 (cf **[Ba] p.15)** *The Cohen forcing notion is Souslin.*

The **Amoeba forcing** was first discussed by Solovay in his quest for proving that $MA(\kappa) \Rightarrow \mathcal{N}$ is κ-additive.

Definition 2.3.10 ([MS] §4 p.167) *The* **Amoeba forcing** *notion, denoted by $\mathbb{A}$, is defined as follows:*

$$p \in \mathbb{A} \leftrightarrow p \subseteq 2^\omega \wedge p \textit{ is open } \wedge \mu(p) < \frac{1}{2}$$

$$p \leq q \leftrightarrow p \subseteq q$$

Fact 2.3.11 ([MS][1] §4 p.168) *1. $\mathbb{A}$ has the ccc.*

[1] The proof can be found in [Ba] p.18 as well.

2. *In every forcing extension by $\mathbb{A}$, the set of Random reals over the ground models has measure zero.*

Fact 2.3.12 ([BJ] p.5 2.3.2) $\mathbb{A}$ *is Souslin.*

In his quest for the invariant proof for Category, Solovay used the **Dominating Forcing** notion.

Definition 2.3.13 ([MS]) *The* **Dominating forcing** *notion, denoted by* $\mathbb{D}$ *is defined as follows:*

$$\mathbb{D} = \{\langle n, f\rangle : n \in \omega, f \in \omega^{\omega}\}$$

where for $\langle n, f\rangle, \langle m, g\rangle \in \mathbb{D}$,

$$\langle n, f\rangle \geq \langle m, g\rangle \Longleftrightarrow n \geq m \wedge f\restriction m = g\restriction m \wedge f \geq g$$

Fact 2.3.14 (cf **[JuBa] Ch.3)** $\mathbb{D}$ *is σ-centered.*

Corollary 2.3.15 $\mathbb{D}$ *is Souslin.*

Proof By the fact above, $\mathbb{D}$ is σ-centered. Thus $\mathbb{D}$ has the ccc. On the other hand - $\mathbb{D}, \leq_{\mathbb{D}}$ are obviously **Borel**. ■

For a clear description (clearer than the reference), and some deeper research on these four mentioned examples, look in [JuBa] Ch.3 §3.1.

Shelah used in [Sh] a forcing notion similar in many ways to $\mathbb{D}$. This forcing notion is called **Amoeba for category** or **Amoeba meager forcing** notion (Shelah calls it "Universal Meager" in his article).

Definition 2.3.16 ([Sh] §4 p.15) 1. *Let $\mathbb{E}$ be the set of pairs (n, T) such that $n \in \omega$, $T \subseteq 2^{<\omega}$ is a perfect tree and the set of branches of T is a meager subset of 2^{ω}.*

2. *For members of $\mathbb{E}$ we let $(n, T) \leq (m, S)$ iff $n \leq m$ and $T\restriction n = S\restriction n$, and $T \subseteq S$ (notice that the restriction is to n and not to m).*

3. *$\langle \mathbb{E}, \leq\rangle$ is called* **Amoeba meager forcing** *(or* **Amoeba forcing for category***).*

Lemma 2.3.17 ([Sh]) 1. *$\langle \mathbb{E}, \leq\rangle$ is σ-centered.*

2. *$V^{\mathbb{E}} \models$ "$\bigcup\{M : M$ is meager and coded in $\mathbf{V}\}$ is meager".*

3. *$\mathbb{E}$ is Souslin,(and in fact ω-Souslin (see [Ju])).*

I will prove the first part of the lemma. The others can be found in [Sh].

PROOF Recalling the definition of a σ-centered partial order (see 2.1.16), we need to show that there is a function $h : \mathbb{E} \to \omega$ such that

$$\forall n \in \omega \forall F \in [\mathbb{E}]^{<\omega} (\forall p \in F(h(p) = n) \Rightarrow \exists q \in \mathbb{E} \forall p \in F(q \geq p)).$$

We will create a countable partition of $\mathbb{E}$ which will satisfy this condition for each class of it (each class has its own n in the above statement). First let us partition the partial order by its first element. Now, fix $n \in \omega$. There are finitely many possibilities of $T \restriction n$. Thus each class has its first element n equal for all the members of the class, and the second element restricted to n is also equal between the elements of the class. Thus there are countably many classes. Take $h : \mathbb{E} \to \omega$ sending each element to its class number. Thus obviously this function satisfies the condition above: Take n, F as above, and assume F is included in the n-th class (i.e $\forall p \in F(h(p) = n)$). Take S to be the union of the second element in all the conditions in F. Then S is still meager, thus $\langle n, S\rangle \in \mathbb{E}$, and then

$$\forall p \in F(p \leq \langle n, S\rangle).$$

Thus h is the needed function. ■

Definition 2.3.18 *We say that a σ-ideal $\mathcal{I}$ is a* **Souslin Ideal**, *if $\mathcal{I}$ is a non-trivial (i.e $\mathbb{R} \notin \mathcal{I}$) Borel ccc absolute σ-ideal.*

2.4 General Notations

Our notation is standard. However, in forcing considerations we keep the convention that *a stronger condition is the greater one.* $\mathfrak{c}$ stands for the cardinality of the continuum. We use the following notations:

F^* The Borel set constructed in a bigger model, according to the Borel code of F (see cf [Je] p.537-540).

A_c is the **Borel** set coded by c.

$\mathbb{A}$ The Amoeba forcing notion.

$\mathbb{B}$ The Random forcing notion.

$\mathbb{C}$ The Cohen forcing notion.

$\mathbb{D}$ The Dominating forcing notion.

$\mathbb{E}$ The Amoeba meager forcing notion.

A^c is the complement of A concerning the current context.

$Ra(V)$ denotes the set of reals random over V.

BC Borel codes

Chapter 3

Ideals & Souslin Forcing

3.1 Introduction

Preliminaries: In the following chapter we will make a survey of the main results in a late work by H. Judah and A. Roslanowski (cf [JuRo]).

We are interested in representing a forcing notion as a quotient Boolean algebra **Borel**$/\mathcal{I}$ for some ccc Borel σ-ideal $\mathcal{I}$. This is interesting because of several reasons. The first (and the main one) is that if we assume (or prove) some additional properties of the ideal $\mathcal{I}$ then we can use the well-developed machinery of $\mathcal{I}$-random reals (cf [Ku2]). The second reason is that we have a nice description of reals, Borel sets etc in extensions via such algebras.

For the rest of our work we will need the following definitions (for more on the basis of the following, see cf [Je] Ch.17):

Definition 3.1.1 *Let $\mathbb{B}$ be a Boolean algebra. Let*

$$S = \{p : p \text{ is an ultrafilter on } \mathbb{B}\}$$

and for every $u \in \mathbb{B}$, let $X_u = \{p : p \in S \wedge \ u \in p\}$. Let

$$\mathcal{F} = \{X_u : u \in \mathbb{B}\}$$

We call the space S with the topology given by the base $\mathcal{F}$ the **Stone space of** *$\mathbb{B}$ and we denote it by $ST(\mathbb{B})$.*

Definition 3.1.2 *A Boolean algebra $\mathbb{B}$ is said to be a* **countably generated complete Boolean algebra** *if there is a countable set $A \subset \mathbb{B}$ such that for each complete Boolean subalgebra $\mathbb{D}$ of $\mathbb{B}$ ($\mathbb{D} \neq \mathbb{B}$), $A \not\subset \mathbb{D}$. The invariant definition goes for* **countably generated Boolean algebra**. *Note: When we refer to a countably generated forcing notion $\mathbb{P}$, we will actually say that the boolean algebra of $\mathbb{P}$ (see def. 3.2.3) is countably generated.*

Definition 3.1.3 *Let X be a topological space. We say that a set A is a* **Baire subset** *of X if A is generated by a countable series of countable unions intersections and complements of basic open sets (Notice the similarity and the difference to the definition of* **Borel** *sets). We notify the set of Baire subsets of X by BAIRE(X).*

To begin with, we have Sikorski's theorem (cf [Si], §31) which says that *every ccc countably generated complete Boolean algebra $\mathbb{B}$ is isomorphic to the quotient algebra* $\mathbf{Borel}(2^\omega)/\mathcal{I}$ *of Borel subsets of the Cantor space modulo some Borel σ-ideal.* The isomorphism can be described as follows. Let $\mathrm{ST}(\mathbb{B})$ be the Stone space of the algebra $\mathbb{B}$, $\mathcal{M}$ be the σ-ideal of meager sets of the space. Then the algebra $\mathbb{B}$ is isomorphic to the quotient $\mathrm{BAIRE}(\mathrm{ST}(\mathbb{B}))/\mathcal{M}$ of Baire subsets of $\mathrm{ST}(\mathbb{B})$ modulo meager sets. Let $u_n \in \mathbb{B}$ be generators of $\mathbb{B}$ (so $[u_n]_\mathcal{M}$ are generators of $\mathrm{BAIRE}(\mathrm{ST}(\mathbb{B}))/\mathcal{M}$; elements of $\mathbb{B}$ are identified with clopen subsets of $\mathrm{ST}(\mathbb{B})$). Define $\phi : \mathrm{ST}(\mathbb{B}) \longrightarrow 2^\omega$ by $\phi(x)(n) = 1$ iff $u_n \in x$ and let

$$f : \mathbf{Borel}(2^\omega) \longrightarrow \mathrm{BAIRE}(\mathrm{ST}(\mathbb{B}))/\mathcal{M} : A \mapsto [\phi^{-1}[A]]_\mathcal{M}.$$

Then f is a σ-epimorphism of the Boolean algebras and hence

$$\mathbb{B} \cong \mathbf{Borel}(2^\omega)/\mathrm{Ker}(f).$$

The ideal $\mathrm{Ker}(f)$ consists of all Borel sets $A \subseteq 2^\omega$ such that the preimage $\phi^{-1}[A]$ is meager in $\mathrm{ST}(\mathbb{B})$.

As both the space $\mathrm{ST}(\mathbb{B})$ and the ideal of meager sets of it have no nice general description, this approach has several disadvantages. In particular it is difficult to describe and to investigate the ideal $\mathrm{Ker}(f)$. Moreover, generally it has none of the properties we could expect - we should keep in mind the ideals of meager and of null subsets of the Cantor space as "positive" examples here. For these reasons we will present another approach.

Proposition 3.1.4 *Suppose that $\mathcal{I}$ is a ccc Borel σ-ideal on 2^ω, $\mathbb{B} = \mathbf{Borel}/\mathcal{I}$ is the quotient (complete) algebra. Let $\dot{r}$ be a $\mathbb{B}$-name for an element of 2^ω such that $[\![s \subseteq \dot{r}]\!] = [[s]]_\mathcal{I}$ (where $[s]$ is the basic set constructed by s and $[[s]]_\mathcal{I}$ is that set modulo $\mathcal{I}$). Then*

1. *If τ is a $\mathbb{B}$-name for an element of 2^ω then there is a Borel function $f : 2^\omega \longrightarrow 2^\omega$ such that $\Vdash_\mathbb{B} f(\dot{r}) = \tau$.*
2. *If $\dot{B}$ is a $\mathbb{B}$-name for a Borel subset of 2^ω then there is a Borel set $A \subseteq 2^\omega \times 2^\omega$ such that $\Vdash_\mathbb{B} \dot{B} = (A)_{\dot{r}}$, where $(A)_x = \{y : (x, y) \in A\}$.*

Proof *1.* Construct inductively Borel sets $A_s \subseteq 2^\omega$ such that for each $s \in 2^{<\omega}$: $A_s = A_{s^\frown 0} \cup A_{s^\frown 1}$, $A_{s^\frown 0} \cap A_{s^\frown 1} = \emptyset$ and $[\![s \subseteq \tau]\!]_\mathbb{B} = [A_s]_\mathcal{I}$. Put $f(x) = y$ if $x \in \bigcap_{n\in\omega} A_{y\restriction n}$ and $f(x) = \bar{0}$ if $x \notin \bigcap_{n\in\omega} \bigcup_{s\in 2^n} A_s$ (This actually means that $x \notin \mathcal{A}_{<>}$). I will show that this f works. Take G $\mathbb{B}$-generic over V.

Now $V[G] \models s \subseteq \tau \implies [A_s]_{\mathcal{I}} \in G \implies \exists s_2([s_2] \subseteq A_s \wedge [[s_2]]_{\mathcal{I}} \in G) \implies V[G] \models s_2 \subseteq \dot{r} \implies V[G] \models \dot{r} \in [s_2] \implies V[G] \models \dot{r} \in A_s$. So τ is the appropriate y for $\dot{r}$.

2. Let $\{C_n : n \in \omega\}$ enumerate the (clopen) basis of 2^ω. If $\dot{B}$ is a name for an open set then we have a name $\dot{U}$ for a subset of ω such that $\Vdash \dot{B} = \bigcup\{C_n : n \in \dot{U}\}$. Let A_n be a Borel set such that $[A_n]_{\mathcal{I}} = [\![n \in \dot{U}]\!]$ and put $A = \bigcup\{A_n \times C_n : n \in \omega\}$. Since $[A_n]_{\mathcal{I}} = [\![\dot{r} \in A_n]\!]$ (by the assumption on $\dot{r}$) we get that $\Vdash (A)_{\dot{r}} = \dot{B}$. Thus we are done for open sets. Next apply easy induction (note that $(\bigcap_n A_n)_x = \bigcap_n (A_n)_x$ and $(\neg A)_x = \neg(A)_x$). ■

3.2 The ideal

This section is devoted to the construction of a single Ideal denoted $\mathcal{I}_{\mathbb{P}}$, which will be the ideal associated with $\mathbb{P}$. We start with some definitions of well known properties (of Forcing notions & Ideals) and advance to the needed properties and the construction of the Ideal.

Definition 3.2.1 *An ideal $\mathcal{I}$ is* **Borel Absolute** *if for a Borel set A_c (c is the Borel code of A_c (*cf *[Je] p.537)) the property of $A_c \in \mathcal{I}$ is absolute. Examples for absolute ideals are the ideals of Meager sets and of Null sets.*

We want the ideal to be a nice one. We want it to be a ccc absolute σ-ideal, to be able to calculate its complexity, and to ensure invariance of it. For some forcing notions we will construct the ideal on the Baire space in a way giving more possibilities to work with in the manner described above.

Definition 3.2.2 *A forcing notion $\mathbb{P}$ is* **countably-1-generated** *if there are conditions $p_n \in \mathbb{P}$ (for $n \in \omega$) such that*

$$(\forall p \in \mathbb{P})(\forall q \in \mathbb{P}, q \perp p)(\exists n \in \omega)(p_n \perp p \ \&\ p_n \not\perp q).$$

In this situation the conditions p_n ($n \in \omega$) are called σ-1-generators of the forcing notion $\mathbb{P}$.

Definition 3.2.3 *$RO(\mathbb{P})$ is the* **regular open algebra** *of $\mathbb{P}$. The elements of $RO(\mathbb{P})$ are the regular open subsets $B \subseteq \mathbb{P}$ (B is regular if $B = internal(\bar{B})$). $B \le C$ iff $B \supseteq C$. The topology on $\mathbb{P}$ is built up by the basis $\{[p] : p \in \mathbb{P}\}$ where $[p] = \{q \in \mathbb{P} : q \ge p\}$.*

Lemma 3.2.4 *Let $\mathbb{P}$ be a partial order. Then $RO(\mathbb{P})$ is a complete boolean algebra and the map $i(p) = internal(\overline{[p]})$, $i : \mathbb{P} \to RO(\mathbb{P})$ is dense embedding. (*cf *[Ku] pp.63-64)*

Corollary 3.2.5 *If $\mathbb{P}$ is countably-1-generated then the Boolean algebra $RO(\mathbb{P})$ is countably generated and each element of $\mathbb{P}$ is the complement (in the algebra $RO(\mathbb{P})$) of the union of a family of generators.*

PROOF Take $p \in \mathbb{P}$. Take $A = \{[p_n] : n \in \omega\ p_n \perp p\}$. Then A is predense (cf [Je2] p.3) above $\neg[p]$. Therefore $\neg[p] = \sum A$. So the second part is done. Now the first part is clear, taking $\{[p_n] : n \in \omega\}$ to be the generating family of $RO(\mathbb{P})$. ■

From the corollary above, it is clear that elements of $RO(\mathbb{P})$ are unions of elements of the form $-\sum\{p_n : n \in u\}$, $u \subseteq \omega$. For an arbitrary Souslin forcing it is not clear that they should be of this form. Many classical ccc countably generated Boolean algebras are determined by countably-1-generated forcing notions. The Random Algebra is determined by the order of closed sets of positive measure in 2^ω. Clearly this order is countably-1-generated. The Amoeba Algebra for measure, the Amoeba Algebra for category, the Hechler forcing and the Eventually Different Real forcing notion can be represented as countably-1-generated orders. Actually we have no example of a ccc Souslin forcing notion (producing one real extension) which is not of this kind. Judah & Roslanowski proposed the following problem:

Problem 3.2.6 *Suppose $\mathbb{P}$ is a ccc Souslin forcing notion such that the algebra $RO(\mathbb{P})$ is countably generated. Can $\mathbb{P}$ be represented as a ccc Souslin countably-1-generated forcing notion?*

In the following considerations we will assume that every forcing notion is *separative*, i.e. if $p, q \in \mathbb{P}$, $p \not\leq q$ then there is $q_0 \geq q$ such that $q_0 \perp p$. This assumption can be easily avoided if we replace (in some places) inequality in $\mathbb{P}$ by that in $RO(\mathbb{P})$ (in which this is obviously the case).

Proposition 3.2.7 *Suppose $\mathbb{P}$ is an atomless ccc countably-1-generated forcing notion. Then there is a mapping $\pi : \omega^{<\omega} \longrightarrow \mathbb{P}$ such that*

1. *for each $s \in \omega^{<\omega}$ the family $\{\pi(s\hat{\ }n) : n \in \omega\}$ is a maximal antichain above $\pi(s)$,*
2. *$\pi(\langle\rangle) = \emptyset_{\mathbb{P}}$ and*
3. *$\mathrm{rng}(\pi)$ is a set of σ-1-generators for $\mathbb{P}$.*

PROOF Let $\langle p_n : n \in \mathbb{P}\rangle \subseteq \mathbb{P}$ be a sequence of σ-1-generators. Construct inductively infinite maximal antichains $\mathcal{A}_n \subseteq \mathbb{P}$ such that

- for each $p \in \mathcal{A}_n$ the set $\{q \in \mathcal{A}_{n+1} : q \geq p\}$ is an infinite maximal antichain above p, and
- $\{q \in \mathcal{A}_n : p_n \leq q\}$ is a maximal antichain above p_n.

Use these antichains to define π in such a way that $\pi[\omega^{n+1}] = \mathcal{A}_n$. ■

The mapping π given by the above proposition (i.e. satisfying 1-3 of 3.2.7) will be called *a basis of* the forcing notion $\mathbb{P}$.

Note that the formula *"a real b encodes a ccc Souslin forcing notion and (a real) π is a basis of it"* is a Π^1_2-formula (The first part is a Π^1_2-formula (see [Ba] p.13)) since if b is a fixed code for a ccc countably-1-generated Souslin forcing notion, then *π is a basis for the forcing notion coded by b* is Π^1_1 (see [JS], [Ju]). Consequently all the notions above are suitably absolute.

Fix a ccc countable-1-generated atomless Souslin forcing notion $\mathbb{P}$ and a basis $\pi : \omega^{<\omega} \longrightarrow \mathbb{P}$ for it. Let b be a real encoding $\mathbb{P}$.

Definition 3.2.8 *1. For a condition $p \in \mathbb{P}$ we define*

$$\phi(p) = \{x \in \omega^{\omega} : (\forall n \in \omega)(\pi(x \restriction n) \not\perp p)\}.$$

2. *A set $A \subseteq \omega^{\omega}$ is* $\mathbb{P}$-small *if there is a maximal antichain $\mathcal{A} \subseteq \mathbb{P}$ such that $A \cap \bigcup\{\phi(p) : p \in \mathcal{A}\} = \emptyset$.*

3. *A set $A \subseteq \omega^{\omega}$ is* $\mathbb{P}$-σ-small *if it can be covered by a countable union of $\mathbb{P}$-small sets. The family of $\mathbb{P}$-σ-small sets will be denoted by $\mathcal{I}_{\mathbb{P}}$.*

From this point on, we will proceed towards showing this ideal to be the one we are looking for.

Proposition 3.2.9 *1. For each $p \in \mathbb{P}$ the set $\phi(p)$ is closed; $\phi(\pi(s)) = [s]$ for each $s \in \omega^{<\omega}$. If $p \leq q$ then $\phi(q) \subseteq \phi(p)$.*

2. *No set $\phi(p)$ (for $p \in \mathbb{P}$) is $\mathbb{P}$-σ-small, and every singleton is $\mathbb{P}$-small.*

3. *$\mathbb{P}$-small sets constitute an ideal, $\mathcal{I}_{\mathbb{P}}$ is a σ-ideal of subsets of ω^{ω}. Every set from $\mathcal{I}_{\mathbb{P}}$ can be covered by a Σ^0_3-set from $\mathcal{I}_{\mathbb{P}}$.*

Proof *1.* It is clear from the definitions of ϕ and π.
2. Let $p \in \mathbb{P}$ and let $\mathcal{A}_n \subseteq \mathbb{P}$ $(n \in \omega)$ be maximal antichains. We want to find $x \in \omega^{\omega}$ such that $(\forall n \in \omega)(\pi(x \restriction n) \not\perp p)$ (i.e $x \in \phi(p)$) and $(\forall n \in \omega)(\exists q \in \mathcal{A}_n)(\forall m \in \omega)(\pi(x \restriction m) \not\perp q)$ (i.e $(\forall n \in \omega)(\exists q \in \mathcal{A}_n)(x \in \phi(q))$). Take $p_0 \in \mathcal{A}_0$ such that $p_0 \not\perp p$ and find $n_0 \in \omega$ so that $\pi(\langle n_0 \rangle) \not\perp (p \vee p_0)$ (i.e., such that $(\exists q \in \mathbb{P})(q \geq p, p_0, \pi(\langle n_0 \rangle)))$. Choose $p_1 \in \mathcal{A}_1$ such that $p_1 \not\perp (p \vee p_0 \vee \pi(\langle n_0 \rangle))$ and let $n_1 \in \omega$ be such that $\pi(\langle n_0, n_1 \rangle) \not\perp (p \vee p_1 \vee p_0 \vee \pi(\langle n_0 \rangle))$ (one can obviously omit the $\pi(\langle n_0 \rangle)$). Continuing in this fashion we will define $x = \langle n_0, n_1, n_2 \ldots \rangle \in \omega^{\omega}$ which will work (just remember that by ϕ's definition, if $\pi(x \restriction m) \not\perp q$ then $(\forall n \leq m)(\pi(x \restriction n) \not\perp q)$, and that we built x such that $\pi(x \restriction m) \not\perp (p \vee \bigvee\{p_n : n \leq m\})$). Thus $\phi(p)$ is not $\mathbb{P}$-σ-small.

Now suppose that $x \in \omega^{\omega}$. To show that the singleton $\{x\}$ is $\mathbb{P}$-small it is enough to prove that the set $\{p \in \mathbb{P} : x \notin \phi(p)\}$ is dense in $\mathbb{P}$ (thus we will have the needed antichain). Given $q \in \mathbb{P}$, take $q_0, q_1 \geq q$ such that $q_0 \perp q_1$ ($\mathbb{P}$ is atomless). There is $s \in \omega^{<\omega}$ with $\pi(s) \perp q_0$ and $\pi(s) \not\perp q_1$ ($rng(\pi)$ is a set of σ-1-generators of $\mathbb{P}$). If $s \subseteq x$ then $x \notin \phi(q_0)$ and we are done. So suppose that

$x{\restriction}\mathrm{lh}(s) \neq s$. Take $q_2 \geq \pi(s), q_1$. Then $\pi(x{\restriction}\mathrm{lh}(s))$ and q_2 are incompatible and consequently $x \notin \phi(q_2)$.

3. To prove the additivity of $\mathbb{P}$-small sets note that if maximal antichains $\mathcal{A}_i \subseteq \mathbb{P}$ $(i = 0, 1)$ witness that sets $A_i \subseteq \omega^\omega$ are $\mathbb{P}$-small then any maximal antichain $\mathcal{A} \subseteq \mathbb{P}$ refining both $\mathcal{A}_0$ and $\mathcal{A}_1$ witnesses that $A_0 \cup A_1$ is $\mathbb{P}$-small. The second part is obvious. The third part is clear from corollary 3.2.5 and from the definition of $\mathbb{P}$-σ-small sets. ■

Definition 3.2.10 *Let $\dot{r} = \dot{r}_\pi$ be the $\mathbb{P}$-name for a real in ω^ω such that for each $s \in \omega^{<\omega}$ we have $\pi(s) \Vdash_{\mathbb{P}} s \subseteq \dot{r}_\pi$.*

For a real $r \in \omega^\omega$ we define $G(r) = \{p \in \mathbb{P} : r \in \phi(p)\}$.

Proposition 3.2.11 *Let N be a transitive model of ZFC* such that b, π and everything relevant is in N.*

1. *If $G \subseteq \mathbb{P}^N$ is a generic filter over N then $G(\dot{r}^G) \cap N = G$.*
2. *Suppose that $x \in \omega^\omega$ is such that for any maximal antichain $\mathcal{A} \subseteq \mathbb{P}$, $\mathcal{A} \in N$ we have $x \in \bigcup_{p\in\mathcal{A}} \phi(p)$. Then $G(x) \cap N$ is a generic filter over N and $\dot{r}^{G(x)} = x$.*

PROOF First note that, in N, b encodes a ccc Souslin forcing notion and π is a basis for it (Π^1_2 formulas are downward absolute for all models of ZFC*). Moreover if $N \models$"*$\mathcal{A}$ is a maximal antichain in $\mathbb{P}$*" then $\mathcal{A}$ is really a maximal antichain of $\mathbb{P}$. Notice that $\mathcal{P}^N = \mathbb{P} \cap N$ and the same concerns $\perp_{\mathbb{P}}$, $\leq_{\mathbb{P}}$ (for all of these, one can see [Ba]).

1. Let us show that $G \subseteq G(\dot{r}^G)$ first. Take $p \in G$. If $p \notin G(\dot{r}^G)$ then there is $n \in \omega$ such that $\pi(\dot{r}^G{\restriction}n)\perp p$ so $\pi(\dot{r}^G{\restriction}n) \notin G$, but since G is a filter in $\mathbb{P} \cap N$, this contradicts $\pi(\dot{r}^G{\restriction}n) \Vdash (\dot{r}^G{\restriction}n) \subseteq \dot{r}$. Thus we have $G \subseteq G(\dot{r}^G)$. Now, if $p \notin G$, $p \in \mathbb{P} \cap N$ then there is $s \in \omega^{<\omega}$ such that $\pi(s)\perp p$ and $\pi(s) \in G$ (π is a basis for $\mathbb{P}$). Consequently $s \subseteq \dot{r}^G$ and $\dot{r}^G \notin \phi(p)$, so $p \notin G(\dot{r}^G)$.

2. As $x \in \phi(p) \Leftrightarrow p \in G(x)$ it is enough to show that $G(x) \cap N$ is a filter (genericity follows from the definition of ϕ and π). For this it suffices to prove that $G(x) \cap N$ contains no pair of incompatible elements. Thus suppose that $p_0, p_1 \in \mathbb{P} \cap N$ are incompatible. Let $\mathcal{A} \in N$ be a maximal antichain in $\mathbb{P}$ such that (in N) for each $p \in \mathcal{A}$

either there is $s \in \omega^{<\omega}$ such that $p \geq \pi(s)$ and $\pi(s)\perp p_0$
or there is $s \in \omega^{<\omega}$ such that $p \geq \pi(s)$ and $\pi(s)\perp p_1$.

By the choice of x we have that $x \in \phi(p)$ for some $p \in \mathcal{A}$. Let $s \in \omega^{<\omega}$ be such that $p \geq \pi(s)$ and $\pi(s)\perp p_0$ (or $\pi(s)\perp p_1$). Then $s \subseteq x$ (since $\pi(s) \not\perp \pi(x{\restriction}\mathrm{lh}(s))$), and $x \notin \phi(p_0)$ (or $x \notin \phi(p_1)$). Consequently either $p_0 \notin G(x)$ or $p_1 \notin G(x)$, and $G(x)$ is a $\mathbb{P}$-generic filter. Now, since $\pi(s) \in G(x) \Rightarrow s \subseteq x$, $\dot{r}^{G(x)} = x$ is clear. ■

Proposition 3.2.12 *1. Let B be a Borel subset of ω^ω. Then*

$$B \notin \mathcal{I}_{\mathbb{P}} \text{ if and only if } (\exists p \in \mathbb{P})((\phi(p)\backslash B) \in \mathcal{I}_{\mathbb{P}}).$$

2. The formula "c is a code for a Borel set belonging to $\mathcal{I}_{\mathbb{P}}$" is Δ^1_2; it is absolute for all transitive models of ZFC.*

Proof *1.* Since $\phi(p) \notin \mathcal{I}_{\mathbb{P}}$ for any $p \in \mathbb{P}$ (by 3.2.9) we easily get that $(\exists p \in \mathbb{P})(\phi(p)\backslash B \in \mathcal{I}_{\mathbb{P}})$ implies $B \notin \mathcal{I}_{\mathbb{P}}$. Suppose now that $B \notin \mathcal{I}_{\mathbb{P}}$. Let c be a real encoding the Borel set B. Let N be a countable transitive model of ZFC* such that $b, c, \pi, \ldots \in N$. Since $B \notin \mathcal{I}_{\mathbb{P}}$ we find a real $x \in B$ such that

$$x \in \bigcap\{\bigcup_{p \in \mathcal{A}} \phi(p) : N \models \mathcal{A} \textit{ is a maximal antichain in } \mathbb{P}^N\}.$$

(we can find such an x by using the technique of the proof of 3.2.9/2). By 3.2.11 we get that $G = G(x) \cap N$ is a $\mathbb{P}^N$-generic filter over N and $\dot{r}^G = x$. As $N[G] \models \dot{r}^G \in B$ we find $p \in \mathbb{P}^N$ such that $N \models p \Vdash \dot{r} \in \sharp c$ (where $\sharp c$ stands for the Borel set coded by c). We claim that

$$(\phi(p)\backslash B) \cap \bigcap\{\bigcup_{q \in \mathcal{A}} \phi(q) : N \models \mathcal{A} \textit{ is a maximal antichain in } \mathbb{P}^N\} = \emptyset.$$

Suppose not and let y be a real from the intersection. As earlier we have that $G' = G(y) \cap N$ is a $\mathbb{P}^N$-generic filter over N, $\dot{r}^{G'} = y$. Since $p \in G(y)$ we get a contradiction to $y \notin B$.

2. For a real a let $\langle (a)_n : n \in \omega \rangle$ be the sequence of reals coded by a. Let $\mathbf{An} = \{a : \langle (a)_n : n \in \omega \rangle \subseteq \mathbb{P}$ is a maximal antichain $\}$. Clearly $\mathbf{An}$ is a Π^1_1-set (see [Ba] p.5). Now
"c is a Borel code for a set from $\mathcal{I}_{\mathbb{P}}$" $\equiv$

$$(\exists a)((\forall n)((a)_n \in \mathbf{An}) \;\&\; (\forall x \in \sharp c)(\exists n \forall m)(x \notin \phi(((a)_n)_m))) \;\&\; c \in \mathrm{BC}$$

The first part of the conjunction is Σ^1_2, the second part is Π^1_1. Hence the formula is Σ^1_2. On the other hand, by *1.*,
"c is a Borel code for a set not belonging to $\mathcal{I}_{\mathbb{P}}$" $\equiv$

$$(\exists p \in \mathbb{P})((\phi(p)\backslash \sharp c) \in \mathcal{I}_{\mathbb{P}}) \;\&\; c \in \mathrm{BC}.$$

Easily the last formula is Σ^1_2 too. Consequently both formulas are Δ^1_2 and this fact is provable in ZFC. As Σ^1_2 formulas are upward absolute (for models of ZFC*) and Π^1_2 formulas are downward absolute (for models of ZFC*) we are done. ■

Before the last corollary, let me redraw our assumptions: $\mathbb{P}$ is a Souslin ccc forcing notion which is separative, atomless, and countably-1-generated.

Corollary 3.2.13 *$\mathcal{I}_{\mathbb{P}}$ is a Borel ccc absolute σ-ideal on ω^ω. The quotient algebra $\mathbf{Borel}(\omega^\omega)/\mathcal{I}_{\mathbb{P}}$ is a ccc complete Boolean algebra. The mapping*

$$\mathbb{P} \longrightarrow \mathbf{Borel}(\omega^\omega)/\mathcal{I}_{\mathbb{P}} : p \mapsto [\phi(p)]_{\mathcal{I}_{\mathbb{P}}}$$

is a dense embedding (so $RO(\mathbb{P}) \cong \mathbf{Borel}(\omega^\omega)/\mathcal{I}_{\mathbb{P}}$). For each Borel code c: $[\![\dot{r} \in \sharp c]\!]_{\mathbb{P}} = [\sharp c]_{\mathcal{I}_{\mathbb{P}}}$ ■

Let us define another Ideal which will be denoted by $\mathcal{I}^0_{\mathbb{P}}$: Let $p_n \in \mathbb{P}$ (for $n \in \omega$) be such that they completely generate the algebra $RO(\mathbb{P})$ (remember 3.2.5). Let $\dot{r}$ be a $\mathbb{P}$-name for a real from 2^ω such that $[\![\dot{r}(n) = 1]\!]_{RO(\mathbb{P})} = p_n$. Now, define the ideal:

$$\mathcal{I}^0_{\mathbb{P}} = \{B \in \mathbf{Borel} : \Vdash_{\mathbb{P}} \dot{r} \notin B\}$$

Corollary 3.2.14 *$\mathcal{I}_{\mathbb{P}} = \mathcal{I}^0_{\mathbb{P}}$.* ■

3.3 Baire Property

The next section is also connected to a work of Judah & Repický in [JuRe].

Definition 3.3.1 *A family $\mathcal{F}$ of subsets of ω^ω is* **a category base on** ω^ω *if $|\mathcal{F}| = \mathfrak{c}$, $\bigcup \mathcal{F} = \omega^\omega$ and for each subfamily $\mathcal{G} \subseteq \mathcal{F}$ of disjoint sets, $|\mathcal{G}| < \mathfrak{c}$ and each $A \in \mathcal{F}$*

if $(\exists B \in \mathcal{F})(B \subseteq A \cap \bigcup \mathcal{G})$ then $(\exists B \in \mathcal{F})(\exists C \in \mathcal{G})(B \subseteq C \cap A)$, and
if $\neg(\exists B \in \mathcal{F})(B \subseteq A \cap \bigcup \mathcal{G})$ then $(\exists B \in \mathcal{F})(B \subseteq A \setminus \bigcup \mathcal{G})$.

Definition 3.3.2 *Let $\mathcal{F}$ be a category base on ω^ω and let $A \subseteq \omega^\omega$.*

1. *A is* **$\mathcal{F}$-singular** *if $(\forall B \in \mathcal{F})(\exists C \in \mathcal{F})(C \subseteq B \setminus A)$.*
2. *A is* **$\mathcal{F}$-meager** *if it can be covered by a countable union of $\mathcal{F}$-singular sets.*
3. *A has* **$\mathcal{F}$-Baire property** *if for every $B \in \mathcal{F}$ there is $C \in \mathcal{F}$ such that $C \subseteq B$ and either $C \cap A$ is $\mathcal{F}$-meager or $C \setminus A$ is $\mathcal{F}$-meager.*

Theorem 3.3.3 (Marczewski, Morgan) *Assume $\mathcal{F}$ is a category base on ω^ω. Then $\mathcal{F}$-meager sets constitute a σ-ideal on ω^ω. Sets with the $\mathcal{F}$-Baire property form a σ-field which is closed under the Souslin operation $\mathcal{A}$.* ■

Suppose that π is a basis for a ccc Souslin forcing $\mathbb{P}$. Assume that

$$(\forall p, q \in \mathbb{P})(p \perp q \text{ if and only if } \phi(p) \cap \phi(q) = \emptyset).$$

Proposition 3.3.4 *If $\pi, \mathbb{P}$ are as above then the family $\mathcal{F}_{\mathbb{P}} = \{\phi(p) : p \in \mathbb{P}\}$ is a category base. The family of $\mathcal{F}_{\mathbb{P}}$-singular sets is the family of $\mathbb{P}$-small subsets of ω^{ω}, $\mathcal{F}_{\mathbb{P}}$-meager sets agree with $\mathcal{P}$-σ-small sets.* ■

Corollary 3.3.5 *Let $\pi, \mathbb{P}$ be as above.*

1. *Sets with $\mathcal{F}_{\mathbb{P}}$-Baire property constitute a σ-field of subsets of ω^{ω}. This σ-field is closed under the Souslin operation $\mathcal{A}$ and contains all Borel sets (and hence it contains both Σ^1_1 and Π^1_1 sets).*
2. *If $A \in \omega^{\omega}$ is a Π^1_1 set then either $A \in \mathcal{I}_{\mathbb{P}}$ or $(\exists p \in \mathbb{P})(\phi(p) \backslash A \in \mathcal{I}_{\mathbb{P}})$.* ■

3.4 Absolute Ideals

Before we conclude this chapter, let me define some notions which will be useful in the chapters to follow, and prove some useful facts about forcing notions of the form $\mathbf{Borel}/\mathcal{I}$ where $\mathcal{I}$ is an absolute σ-ideal.

To the rest of the rest of our work we will use the following context as our main point of view: Let $\mathbb{P} = \mathbf{Borel}(\omega^{\omega})/\mathcal{I}$ be a Souslin ccc forcing notion ($\mathcal{I}$ is a σ-ideal on ω^{ω}). Notice that this context is quite similar to the one used in [Ku2] with the difference that Kunen requires that his ideal (which he calls "reasonable ideal") will have a form of the Fubini property. We do not need that property, and we do not require it (although in some cases (e.g., $\mathbb{L} \models \neg\Delta^1_2(\mathbb{P})$ in §4.3) it would have made things much easier.

Definition 3.4.1 *Let $n \geq 1$. $\Sigma^1_n(\mathcal{I})$ is the following statement:*
For every Σ^1_n subset of the real line A, there is a **Borel** *set B such that $B \triangle A \subseteq \mathcal{I}$. (note that we abuse notations by referring to $\mathcal{I}$, which is defined on the* **Borel** *σ-algebra, as its expansion to the real line.) We say also, that a set A has the* $\mathbb{P}$**.p (**$\mathbb{P}$**-property)***, if there is a* **Borel** *set B such that $B \triangle A \subseteq \mathcal{I}$.*

We want to show that the basic facts we need about Random and Cohen forcing, apply also to our general case. We will prove the following two lemmas using the absoluteness of the ideal $\mathcal{I}$. For the Random/Cohen case one can look at [Je] pages 542 through 544. For the definition and main facts about Borel Codes, one may use [Je] Ch.42 pp.537-540 and also [Ku2] for basic facts.

Lemma 3.4.2 *Let $\mathcal{M}$ be a transitive model of ZFC. If G is an $\mathcal{M}$-generic filter on $\mathbb{P}$, then there is a unique real number x_G such that for all $B \in \mathbb{P}$*

$$x_G \in B^* \Leftrightarrow [B]_{\mathbb{P}} \in G \tag{3.1}$$

The formula (3.1) determines G and hence $\mathcal{M}[G] = \mathcal{M}[x_G]$.

PROOF To start, we claim that there is at most one real number x that satisfies

$$\forall B \in \mathbf{Borel}(x \in B^* \Leftrightarrow [B] \in G). \tag{3.2}$$

If x satisfies (3.2), then x belongs to all B^* such that $[B] \in G$. If $x < y$ are two real numbers, let r be a rational number such that $x < r < y$, and let A be the interval $(r, \infty) \subseteq \mathbb{R}$. Either $[A]$ or $[\mathbb{R} \setminus A]$ belong to G, but $x \notin A^*$ and $y \notin (\mathbb{R} \setminus A)^*$.

In order to show that there exists a real number x that satisfies (3.2), let

$$x = sup\{r : r \text{ is a rational number and } [(r, \infty)] \in G\}. \tag{3.3}$$

By the genericity of G, there exists r such that $[(r, \infty)] \notin G$, and hence the supremum (3.3) exists. Note also that $x \notin \mathcal{M}$ (by the genericity of G). We shall show that x satisfies (3.2). We shall show, by induction on **Borel** codes in $\mathcal{M}$, that for every $c \in BC^{\mathcal{M}}$,

$$x \in A_c^* \iff [A_c] \in G. \tag{3.4}$$

First we consider the Σ_1^0-codes (in $\mathcal{M}$), and let us start with those $c \in \Sigma_1^0 \cap \mathcal{M}$ that code a rational interval, i.e., such that $c(n) = 1$ for exactly one n; then c codes the interval I_n. Let $I_n = (p, q)$. We have

$$\begin{aligned} x \in A_c^* &\iff p < x < q \\ &\iff p < sup\{r : [(r, \infty)] \in G\} < q \\ &\iff [(p, \infty)] \in G \wedge [(q, \infty)] \notin G \\ &\iff [(p, q)] \in G \iff [A_c] \in G. \end{aligned}$$

Now, if $c \in \Sigma_1^0$, then $A_c = \bigcup_{n=0}^{\infty} I_{k_n}$, where $\{k_n : n = 0, 1, \ldots\}$ is the set $\{k : c(k) = 1\}$, and we have

$$\begin{aligned} x \in A_c^* &\iff x \in \bigcup_{n=0}^{\infty} I_{k_n}^* \\ &\iff \exists n (x \in I_{k_n}^*) \\ &\iff \exists n ([I_{k_n}] \in G) \\ &\iff \textstyle\sum_{n=0}^{\infty} [I_{k_n}] \in G \\ &\iff [\bigcup_{n=0}^{\infty} I_{k_n}] \in G \iff [A_c] \in G. \end{aligned}$$

Next let $\alpha < \omega_1^{\mathcal{M}}$ and let $c \in \Pi_\alpha^0 \cap \mathcal{M}$, and let us assume that (3.4) holds for all $c \in \Sigma_\alpha^0 \cap \mathcal{M}$. We may assume that $c(0) = 0$; then $u(c) \in \Sigma_\alpha^0 \cap \mathcal{M}$ and $A_{u(c)} = \mathbb{R} \setminus A_c$, and we have

$$x \in A_c^* \iff x \notin A_{u(c)}^* \iff [A_{u(c)}] \notin G \iff [A_c] \in G.$$

Finally, the induction step for Σ_α^0 is handled in a way similar to the case for $c \in \Sigma_\infty^0$. Thus (3.4) holds for every $c \in BC^{\mathcal{M}}$, and thus x is the unique real number that satisfies (3.1). ■

One should notice that in fact we did not use the absoluteness of the ideal, but just the structure of the partial order (being of the form $\mathbf{Borel}/\mathcal{I}$). The following lemma provides a characterization of $\mathbb{P}$-reals.

Lemma 3.4.3 *A real number is a $\mathbb{P}$-real over $\mathcal{M}$ if and only if it does not belong to any Borel set $I \in \mathcal{I}$ with a code in $\mathcal{M}$.*

PROOF On the one hand, if x is a $\mathbb{P}$-real over $\mathcal{M}$, let G be an $\mathcal{M}$-generic filter on $\mathbb{P}$ such that $x = x_G$. Then if $A_c \in \mathcal{I}$ then $[A_c] \notin G$, and by 3.4.2, $x \notin A_c^*$.

On the other hand, let x be such that $x \notin A_c^*$ whenever $A_c \in \mathcal{I}$ (and $c \in \mathcal{M}$). First we observe that if $[A_c] = [A_d]$ then $A_c \triangle A_d \in \mathcal{I}$, hence $A_c^* \triangle A_d^* \in \mathcal{I}^*$ (by the absoluteness of $\mathcal{I}$). It follows that x belongs to A_c^* if and only if x belongs to A_d^*. Let

$$G = \{[A_c] : x \in A_c^*\}. \tag{3.5}$$

It is easy to see that G is a filter on $\mathbb{P}$: If $[A_c] \in G$ and $[A_d] \in G$, then $x \in A_c^* \cap A_d^*$ and hence $[A_c \cap A_d] \in G$; similarly, if $[A_c] \geq [A_d]$ and $[A_c] \in G$, then $[A_d] \in G$ (recall that we use "$p \geq q$" to denote "p is stronger than q"). We shall show that G is $\mathcal{M}$-generic. Since $\mathbb{P}$ satisfies the ccc, it suffices to show that if $\{A_{c_n} : n \in \omega\} \in \mathcal{M}$ is such that $\sum_{n=0}^{\infty}[A_{c_n}] \in G$, then some $[A_{c_n}]$ is in G. But this is true because

$$\sum_{n=0}^{\infty}[A_{c_n}] = [\bigcup_{n=0}^{\infty} A_{c_n}] \text{ and } (\bigcup_{n=0}^{\infty} A_{c_n})^* = \bigcup_{n=0}^{\infty} A_{c_n}^*.$$

Finally, we claim that $x = x_G$. But this follows from (3.5), by the genericity of G. Thus a real number x is a $\mathbb{P}$-real over $\mathcal{M}$ if and only if $x \notin A_c^*$ for any **Borel** set $A_c \in \mathcal{I}^{\mathcal{M}}$. ■

Corollary 3.4.4

$$Pr(\mathcal{M}) = \mathbb{R}^* \setminus \bigcup\{A_c^* : c \in BC^{\mathcal{M}} \wedge A_c \in \mathcal{I}\}.$$

PROOF Notice that by the last lemma we get

$$Pr(\mathcal{M}) = \mathbb{R}^* \setminus \bigcup\{A_c^* : c \in BC^{\mathcal{M}} \wedge A_c^* \in \mathcal{I}\}.$$

Thus by the absoluteness of $\mathcal{I}$ we get

$$Pr(\mathcal{M}) = \mathbb{R}^* \setminus \bigcup\{A_c^* : c \in BC^{\mathcal{M}} \wedge A_c \in \mathcal{I}\}.$$

■

In the rest of our work we will abuse notations and use the notations of subsets of the plain, also for their class in $\mathbb{P}$ (modulo the ideal $\mathcal{I}$). For example B will also denote $[B]_{\mathbb{P}}$.

Chapter 4

Examples

4.1 Introduction

Recalll that $\mathbb{P} = \mathbf{Borel}(\omega^\omega)/\mathcal{I}$ is a Souslin ccc forcing notion (I is a σ-ideal on ω^ω) (see 3.4).

We want to have an intuition on the connections that can be established between Souslin absoluteness, Uniformization, and regularity properties for projective sets. We will do so by giving some examples, and investigate these examples through the above criterions. The first will be the well known model of solovay.

4.2 Solovay's model

We will refer the reader mostly to [Je] but the reader may as well look at the original paper [So], or at the new ideas in [JuBa] Ch.9 §6.

To get a model where all projective sets are measurable, Solovay started with the constructible universe $\mathbb{L}$ and an inaccessible cardinal $\kappa \in \mathbb{L}$. The forcing extension, where all projective sets are measurable and have the property of Baire, was obtained by collapsing κ onto $\aleph_1$. The Solovay model is known to approve with the following theorem:

Theorem 4.2.1 (Solovay in [So]) *Assume that there exists an inaccessible cardinal.*

1. *There is a model of $ZF+DC$ in which all sets of real numbers are Lebesgue measurable and have the property of Baire, and every uncountable set of reals has a perfect subset.*
2. *There is a model V of ZFC such that $V \models \forall n \in \omega(\Sigma^1_n(L)\&\Sigma^1_n(B))$ and every uncountable projective set contains a perfect subset.*

We will prove that Solovay's model also satisfies the same properties for forcing notions of the form $\mathbb{P} = \mathbf{Borel}/\mathcal{I}$, where $\mathcal{I}$ is an absolute σ-ideal (see 3.4).

4.2.1 Solovay's sets of reals

Let $\mathcal{M}$ be a transitive model of ZFC.

Definition 4.2.2 (cf **[Je] Ch.42 p.544**) *Let S be a set of reals. We say that the set S is* **Solovay over** $\mathcal{M}$ *if there is a formula $\varphi(x)$ with parameters in $\mathcal{M}$, such that*

$$\forall x \in \mathbb{R}(x \in S \Longleftrightarrow \mathcal{M}[x] \models \varphi(x)) \tag{4.1}$$

Lemma 4.2.3 *Let S be a Solovay set of reals over $\mathcal{M}$. There is a Borel set A such that*

$$S \cap Pr(\mathcal{M}) = A \cap Pr(\mathcal{M})$$

Proof Let us consider the forcing language in $\mathcal{M}$ associated with $\mathbb{P}$. Let G be the canonical name for a generic ultrafilter on $\mathbb{P}$, and let a be the canonical name for a $\mathbb{P}$-real. Let $\varphi(x)$ be a formula with parameters in $\mathcal{M}$ such that (4.1) holds for all x. Let $A_c \in \mathbb{P}$ such that

$$[A_c] = [\![\varphi(a)]\!]$$

and let $A = A_c^*$. The set A is a Borel set (in the universe). I claim that for all $x \in Pr(\mathcal{M})$, $x \in S \Leftrightarrow x \in A$. But if x is a $\mathbb{P}$-real over $\mathcal{M}$, let G be $\mathcal{M}$-generic on $\mathbb{P}$ such that $x = a_G$, so a is a name for x and we have

$$\begin{array}{lllll} x \in S & \Leftrightarrow & \mathcal{M}[x] \models \varphi(x) & \Leftrightarrow & \mathcal{M}[G] \models \varphi(x) \\ \Leftrightarrow & [\![\varphi(a)]\!] \in G & \Leftrightarrow & [A_c] \in G & \Leftrightarrow \quad x \in A_c^* \end{array}$$

Thus A is the required set. ■

4.2.2 The Lévy collapsing forcing notion

We will now define the Lévy Collapsing forcing notion, and study some of its properties:

Definition 4.2.4 (cf **[JuBa] Ch.9 p.287 & [Je] Ch.19 p.182**) *Let μ be a regular cardinal and let λ be an ordinal such that $\lambda > \mu$. Let $\mathbb{P}_\lambda = \lambda^{<\mu}$ with $p_\alpha < q_\alpha \Leftrightarrow p_\alpha \subset q_\alpha$. Let $Coll(\mu, < \lambda)$ be the product with $< \mu$-support of $\langle \mathbb{P}_\alpha; \alpha < \lambda \rangle$. This partial order is called the* **Lévy Collapse**. *Note that*

$$p \in Coll(\mu, < \lambda) \Leftrightarrow p \in \prod_{\alpha < \lambda} \mathbb{P}_\alpha \ \wedge \ |supp(p)| < \mu$$

($supp(p) = \{\alpha : p(\alpha) \neq 0\}$). For $p, q \in Coll(\mu, < \lambda)$,

$$p \leq q \Leftrightarrow \forall \alpha < \lambda(p(\alpha) \leq q(\alpha)).$$

Let κ be an inaccessible cardinal in $\mathcal{M}$. We will concentrate on $Coll(\aleph_0, < \kappa)$ (i.e finite functions under a product with finite support).

Lemma 4.2.5 *1. $Coll(\aleph_0, < \kappa)$ is κ-cc.*

2. $\Vdash_{Coll(\aleph_0,<\kappa)}$ "$\kappa = \aleph_1$"

PROOF 1.: Take $\{p_\alpha : \alpha < \kappa\} \subseteq Coll(\aleph_0, < \kappa)$. By the Δ-system-lemma (cf [Ku] ch.2 §1), there is a set $B \subseteq \kappa$ such that $|B| = \kappa$, and $\{supp(p_\alpha) : \alpha \in B\}$ forms a Δ-system with some root R. Since κ is regular and there are less than κ possibilities for the $(p_\alpha \restriction R)$, there is a $C \subseteq B$ such that $|C| = \kappa$ and $\forall \alpha, \beta \in C((p_\alpha \restriction R) = (p_\beta \restriction R))$. Thus

$$\forall \alpha, \beta \in C(p_\alpha \not\perp p_\beta).$$

2.: Suppose G is $Coll(\aleph_0, \kappa)$-generic over V. By 1., κ remains a cardinal in $V[G]$ (cf [Ku] ch.7 §6 p.213). Fix $\alpha < \kappa$. For every $\gamma < \alpha$ and every $n \in \omega$, the set

$$D_{\gamma,n} = \{p \in Coll(\aleph_0, < \kappa) : n \in dom(p(\alpha)) \wedge \gamma \in ran(p(\alpha))\}$$

is dense in $Coll(\aleph_0, < \kappa)$. Hence $\bigcup_{p\in G} p(\alpha)$ maps ω onto α. ■

Lemma 4.2.6 (cf **[Je] Ch.25 p.280**) *Let G be a generic filter on the Lévy collapse $\mathbb{P}$, and let X be a countable set of ordinals in $\mathcal{M}[G]$. Then there exists an $\mathcal{M}[X]$-generic filter H on $\mathbb{P}$ such that $\mathcal{M}[X][H] = \mathcal{M}[G]$.*

Theorem 4.2.7 (cf **[Je] Ch.25 p.281**) *The Lévy algebra B is homogeneous, in the following sense: If A and A' are isomorphic complete subalgebras of B such that $|A| = |A'| < |B|$ and if π_0 is an isomorphism between A and A', then there exists an automorphism π of B such that $\pi(a) = \pi_0(a)$ for all $a \in A$.*

Lemma 4.2.8 (cf **[Je] Ch.25 p.282**) *Let B be the Lévy algebra (in a ground model $\mathcal{N}$), and let C be a complete subalgebra of B such that $|C| < |B|$. For any formula $\varphi(x)$, if x is a C-valued name, i.e, $x \in \mathcal{N}^C$, then*

$$[\![\varphi(x)]\!]_B \in C$$

4.2.3 Solovay's model

Now that we have defined the Lévy collapsing forcing notion, let $\mathbb{P} = Coll(\aleph_0, < \kappa)$ and let B be the Lévy algebra for $\mathbb{P}$ (i.e $B = RO(\mathbb{P})$). Let G be an $\mathcal{M}$-generic ultrafilter on B. We shall show that $\mathcal{M}[G] \models \forall n \in \omega(\Sigma^1_n(\mathcal{I}))$

Lemma 4.2.9 *Let $s \in \mathcal{M}[G]$ be an infinite sequence of ordinals. The set of all reals (in $\mathcal{M}[G]$) that are not $\mathbb{P}$-reals over $\mathcal{M}[s]$ is in the ideal $\mathcal{I}$.*

PROOF Since the algebra B is κ-saturated (by 4.2.5), there exists a subalgebra $D \subseteq B$ such that $|D| < \kappa$ and $\mathcal{M}[s] = \mathcal{M}[D \cap G]$ (where $\mathcal{M}[D \cap G]$ is the extension of $\mathcal{M}$ by the filter $D \cap G$ in the algebra D). It follows that κ is inaccessible in $\mathcal{M}[s]$ (since $|D| < \kappa$), and since $\kappa = \aleph_1^{\mathcal{M}[G]}$, $\mathcal{M}[s]$ has only countably many subsets of ω (D is also κ-saturated). Thus there are only countably many Borel codes in $\mathcal{M}[s]$, and since every $\mathbb{P}$-real is not in any set in the ideal $\mathcal{I}$, the set $Pr(\mathcal{M}[s])^c$ is a union of countably many sets in the ideal, thus $Pr(\mathcal{M}[s])^c \in \mathcal{I}$. ■

Lemma 4.2.10 *Let $X \in \mathcal{M}[G]$ be a set of reals that is definable in $\mathcal{M}[G]$ from a sequence s of ordinals. Then X is (in $\mathcal{M}[G]$) Solovay over $\mathcal{M}[s]$.*

PROOF We shall first prove the following: Given a formula φ, there is a formula $\tilde{\varphi}$ such that for every sequence of ordinals $x \in \mathcal{M}[G]$

$$\mathcal{M}[G] \models \varphi(x) \iff \mathcal{M}[x] \models \tilde{\varphi}(x).$$

Recall that $\mathbb{P}$ is the Lévy forcing notion for κ. The forcing conditions are finite, and thus the definition of $\mathbb{P}$ is absolute for all models. If $\mathcal{N}$ is a model, we denote $\mathcal{N}^{\mathbb{P}}$ the boolean valued model constructed in $\mathcal{N}$ using $\mathbb{P}$. Let $\tilde{\varphi}(x)$ be the following formula

$$[\![\tilde{\varphi}(\check{x})]\!]^{\mathcal{M}[x]} = 1$$

Let x be a countable sequence of ordinals in $\mathcal{M}[G]$. We shall show that

$$\mathcal{M}[G] \models \varphi(x) \iff \mathcal{M}[x] \models \tilde{\varphi}(x)$$

By 4.2.6 there exists an $\mathcal{M}[x]$-generic filter H on $\mathbb{P}$ such that $\mathcal{M}[G] = \mathcal{M}[x][H]$. Arguing in $\mathcal{M}[x]$, we invoke the homogeneity of the Lévy algebra (see 4.2.8): the Boolean value $b = [\![\varphi(\check{x})]\!]^{\mathcal{M}[x]}$ is either 0 or 1. Since H is generic on $\mathbb{P}$ over $\mathcal{M}[x]$, $\varphi(x)$ is true in $\mathcal{M}[x][H]$ if $b = 1$, and false if $b = 0$. Hence $\varphi(x)$ is true in $\mathcal{M}[G]$ if and only if $\tilde{\varphi}(x)$ is true in $\mathcal{M}[x]$. $\mathbb{P}$) of $\psi(z)$

Clearly the above argument works also for a formula φ with two variables: There is a $\tilde{\varphi}$ such that for all $x, y \in \mathcal{M}[G] \cap Ord^{\omega}$:

$$\mathcal{M}[G] \models \varphi(x, y) \iff \mathcal{M}[x, y] \models \tilde{\varphi}(x, y).$$

Now, let $X \in \mathcal{M}[G]$ be a set of reals that is definable in $\mathcal{M}[G]$ from a sequence of ordinals s. For some formula φ

$$x \in X \iff \mathcal{M}[G] \models \varphi(x, s)$$

For all reals $x \in \mathcal{M}[G]$. Since every real can be concidered a countable sequence of ordinals, we have, for all $x \in \mathbb{R}^{\mathcal{M}[G]}$

$$x \in X \iff \mathcal{M}[s, x] \models \tilde{\varphi}(x, s) \iff \mathcal{M}[s][x] \models \tilde{\varphi}(x, s).$$

Thus X is Solovay over $\mathcal{M}[s]$. ■

Corollary 4.2.11 *In $\mathcal{M}[G]$, every set of reals A, definable from a sequence of ordinals (and in particular, every projective set of reals), has a corresponding Borel set B such that $B \triangle A \subseteq \mathcal{I}$.*

PROOF Assume A is a set as mentioned above. Take s the corresponding sequence of ordinals. By 4.2.10 A is Solovay over $\mathcal{M}[s]$, thus by 4.2.3 there is a Borel set B such that $A \cap Pr(\mathcal{M}) = B \cap Pr(\mathcal{M})$. Now, since by 4.2.9 the set of all reals that are not $\mathbb{P}$-reals is in the ideal $\mathcal{I}$, we have $A \triangle B \in \mathcal{I}$. ■

4.3 The Constructible Universe

In the following section, we will prove that for each souslin ideal $\mathcal{I} \in \mathbb{L}$ there is a Δ^1_2-set A such that $\mathbb{L} \models$ "A does not have the Baire property concerning $\mathcal{I}$".

The general and easy way of showing that property concerning the ideal of Null sets and the ideal of Meager sets, is a well known use of the Fubini theorem (see [Je] Ch.41 p.527). We will use another method which is more difficult but will suit our general case. The development of this proof to the ideals of Null and Meager sets is described in [Mo] (2H.10 p.112, 5A.7 p.280, 5A.8 p.281).

Definition 4.3.1 (cf **[Je] Ch.1 p.33)** *A set of reals is* **perfect** *if it is closed, nonempty, and has no isolated points.*

Definition 4.3.2 ([Mo] §4F p.247) *A set $P \subseteq \mathbb{R}$ is* **thin** *if P has no perfect subsets.*

The following definition corresponds to the definition of μ-measurable (cf [Mo] §2H p.112).

Definition 4.3.3 *$f : \omega^\omega \to \omega^\omega$ is* **$\mathcal{I}$-Souslin** *if for each open set U in ω^ω, $f^{-1}(U)$ is $\mathcal{I}$-Souslin (has the $\mathbb{P}$.p).*

Definition 4.3.4 ([Mo] §5A p.279) *Let $f : \omega^\omega \to \omega^\omega$. The* **Graph** *of f (denoted $Graph(f)$), is*

$$Graph(f) = \{(x, y) : f(x) = y\}$$

(This is actually the function as a set in $(\omega^\omega)^2$)

Definition 4.3.5 (cf **[Je] p.114)** *Let $f : \omega^\omega \to \omega^\omega$. We say that f is a* **Borel function** *if $Graph(f)$ is Borel (We will not prove it here, but this definition is equivalent to being a Borel function in the topological sense (Borel Measurable, see [Mo])).*

Definition 4.3.6 *$f : B \to \omega^\omega$ is B-**continuous** ($B \subseteq \omega^\omega$), if there is a Borel function $f^* : \omega^\omega \to \omega^\omega$ such that*

$$\forall x \in B \quad f(x) = f^*(x).$$

(On ω^ω this is equivalent to being a Borel Measurable function (see [Mo]). We do not need this fact though.)

Lemma 4.3.7 *Assume $f : \omega^\omega \to \omega^\omega$ is $\mathcal{I}$-Souslin. Then there exists a Borel set $B \subseteq \omega^\omega$ which is the complement of a set in the ideal (i.e $B^c \in \mathcal{I}$), and such that the restriction $f \restriction B$ of f to B is B-continuous.*

PROOF For each $s \in Seq$ let G_s be Borel in ω^ω such that $f^{-1}([s]) \triangle G_s \in \mathcal{I}$ ($[s]$ is the basic clopen set created by s). We can Choose such a G_s, since $f^{-1}([s])$ is a Σ^1_1 set. Choose a Borel set $Q_s \in \mathcal{I}$ such that

$$f^{-1}([s]) \triangle G_s \subseteq Q_s$$

(We can take such a set Q_s, since the σ-ideal $\mathcal{I}$, is Souslin (see 2.3.18)). Take $Q = \cup_{s \in Seq} Q_s$. Now $B = Q^c$. We will show that $f \restriction B$ is B-continuous.

Let us enumerate Seq by $\langle s_n ; n \in \omega \rangle$ which preserves precedense by lh (i.e $\forall n \in \omega(\mathrm{lh}(s_n) \leq \mathrm{lh}(s_{n+1})))$. We can obviously take this enumeration to be Borel. For each $n \in \omega$ Take

$$\begin{aligned} f_0(x) = &\quad \langle\rangle \\ f_{n+1}(x) = &\quad \left\{ \begin{array}{ll} s_{n+1} & x \in G_{s_{n+1}} \\ f_n(x) & else \end{array} \right\} \end{aligned}$$

For each $n \in \omega$, f_n is Borel. Define

$$f^* = \lim_{n \to \infty} f_n(x).$$

f^* is obviously Borel. Now, take $x \in B$. Take $\epsilon > 0$. Then

$$\exists n \in \omega (x \in [s_n] \wedge |x - s_n| < \epsilon).$$

Thus

$$\forall x \in B \quad f(x) = f^*(x).$$

■

Lemma 4.3.8 (cf [Mo] §2C p.81) *Every uncountable Σ^1_1 set has a prefect subset.*

Lemma 4.3.9 *Assume $f : \omega^\omega \to \omega^\omega$ and $Graph(f)$ is thin. Then f is not $\mathcal{I}$-Souslin.*

PROOF By lemma 4.3.7, if f is $\mathcal{I}$-Souslin then there is a Borel set B such that $B^c \in \mathcal{I}$, and $f \restriction B$ is B-continuous. Now B is uncountable (elsewhere $B \in \mathcal{I}$ and $B^c \in \mathcal{I}$ (by the choice of B) and thus $\mathbb{R} \in \mathcal{I}$ - Contradiction), so the injective image

$$B^* = \{(x, f(x)) : x \in B\}$$

is also uncountable, **Borel** and a subset of $Graph(f)$ (f is B-continuous on B). But then B^* must have a perfect subset by 4.3.8, contradicting the hypothesis. ■

Lemma 4.3.10 ([Mo] 5A.6 p.279) *Assume $\omega^\omega \subseteq \mathbb{L}$ (e.g $V = \mathbb{L}$). Then there exists a function $f : \omega^\omega \to \omega^\omega$ whose graph $Graph(f)$ is Π^1_1 and thin.*

PROOF Let $\leq_{\mathbb{L}}$ be a Σ^1_2-good wellordering of ω^ω of rank $\aleph_1$, and put

$$P(\alpha, \beta) \iff \alpha \leq_{\mathbb{L}} \beta \wedge \beta \in WO \wedge (\forall \gamma <_{\mathbb{L}} \beta) \neg(\gamma \in WO \wedge |\gamma| = |\beta|)$$

where WO is the set of ordinal codes (see [Mo] §4A). Clearly P is Σ^1_2, so let

$$P(\alpha, \beta) \iff (\exists \gamma) Q(\alpha, \beta, \gamma)$$

with Q in Π^1_1. considering Q as a subset of $\omega^\omega \times (\omega^\omega \times \omega^\omega)$, let Q^* uniformize Q in Π^1_1, so that for each α,

$$(\exists \beta)(\exists \gamma) Q(\alpha, \beta, \gamma) \iff (\exists \beta)(\exists \gamma) Q^*(\alpha, \beta, \gamma)$$

and

$$Q^*(\alpha, \beta, \gamma) \wedge Q^*(\alpha, \beta', \gamma') \Rightarrow \beta = \beta' \wedge \gamma = \gamma'.$$

By (cf [Mo] §4A.6) Q^* has no perfect subset. Since Q^* is obviousely the graph of the function

$$f : \omega^\omega \to \omega^\omega \times \omega^\omega$$

this proves the result for a function from ω^ω to $\omega^\omega \times \omega^\omega$, from which the general fact follows by taking Δ^1_1 isomorphism and using ([Mo] 4F.7). ■

Theorem 4.3.11 *Assume $\omega^\omega \subseteq \mathbb{L}$. Then there is a Δ^1_2 set which is not $\mathcal{I}$-Souslin (has the $\mathbb{P}$.p).*

PROOF Take $f : \omega^\omega \to \omega^\omega$ as in 4.3.10. Now, we will show that for each $p, q \in \mathbb{Q}$, the set $A_{p,q} = \{x : p < f(x) < q\}$ is a Δ^1_2 set.

$$f(x) < q \iff \exists y(((x, y) \in Graph(f)) \wedge y < q).$$

Thus, "$f(x) < q$" is a Σ^1_2 sentence. The same goes for "$p < f(x)$". on the other hand "$q > f(x)$" and "$f(x) > p$" are also Σ^1_2 sentences, thus "$x \in A_{p,q}$" is a Δ^1_2. This is true for all $p, q \in \mathbb{Q}$. Finally we will show that, if for all $p, q \in \mathbb{Q}$, $A_{p,q}$

is $\mathcal{I}$-Souslin (has the $\mathbb{P}.p$), then f is $\mathcal{I}$-Souslin. Take U open in ω^ω. Then U is a countable union of sets of the form (p, q). Thus, since $A_{p,q} = f^{-1}((p, q))$, we get that $f^{-1}(U)$ is a countable union of $\mathcal{I}$-Souslin sets, thus $\mathcal{I}$-Souslin. Thus in that case f is $\mathcal{I}$-Souslin, contradicting 4.3.9. Thus

$$\exists p, q \in \mathbb{Q}(A_{p,q} \text{ is not } \mathcal{I}\text{-Souslin (does not have the } \mathbb{P}.p)).$$

■

Corollary 4.3.12 $\mathbb{L} \models \exists A \in \Delta^1_2(A$ *is not* $\mathcal{I}$*-Souslin*). ■

Chapter 5

Souslin Absoluteness *vs* Uniformization

5.1 General Facts

Lemma 5.1.1 *Let σ be a $\mathbb{P}$-name for a real number. Then there is a* **Borel** *function f such that for a $\mathbb{P}$-real a over V,*

$$V[a] \models \sigma[a] = f(a)$$

PROOF We define f by approximating it using simple functions. We work in $[0,1]$. Let $A_{i,n} = [\![\sigma \in (\frac{i}{2^n}, \frac{i+1}{2^n})]\!]$, $i < 2^n$. Let

$$f_n(x) = \sum_{i<2^n} \frac{i}{2^n} \chi_{A_{i,n}}(x)$$

where $\chi_{A_{i,n}}$ is the characteristic function on $A_{i,n}$. So, each f_n is a simple **Borel** function. Let

$$f(x) = \lim_{n \to \infty} f_n(x)$$

Since $f(x) = y \;\Leftrightarrow\; \forall n \exists m \forall k \geq m(\mid y - f_k(x) \mid < \frac{1}{n})$, f is **Borel**. Now, let a be a $\mathbb{P}$-real over V. Pick $\varepsilon > 0$. For every n, there is a unique $i < 2^n$ such that $a \in A_{i,n}$. But if $a \in A_{i,n}$, $\sigma[a] \in (\frac{i}{2^n}, \frac{i+1}{2^n})$. Also $f_n(a) = \frac{i}{2^n}$. Hence, $\mid \sigma[a] - f_n(a) \mid < \frac{1}{2^n}$. Thus, we can find n such that $\mid \sigma[a] - f_n(a) \mid < \varepsilon$. ∎

Lemma 5.1.2 *Let $n \geq 2$. Assume $\varphi(x)$ is a Π^1_n-formula and f is a* **Borel** *function (Graph(f) is* **Borel***). Then $\varphi(f(x))$ is also a Π^1_n-formula in the additional parameter, borel code for f.*

PROOF Saying that for x, $V \models \varphi(f(x))$ holds, is equivalent to saying

$$V \models (\forall x \exists y(\langle x, y\rangle \in Graph(f))) \wedge (\forall y(\langle x, y\rangle \in Graph(f) \Rightarrow \varphi(y))).$$

■

5.2 Souslin Uniformization

In the following (unless elsewhere is mentioned) we will use the following notions: Let $\mathbb{P} = \mathbf{Borel}(\omega^{\omega})/\mathcal{I}$ be a Souslin ccc forcing notion (I is a σ-ideal on ω^{ω}). μ is the Lebesgue measure (on the appropriate field, concerning the context). One can take $\mathbb{P}$ to be the Random/Cohen forcing notions as an example to the definitions and theorems to follow.

We will show that there is a strong relationship between Uniformization and Souslin absoluteness.

Definition 5.2.1 *Let $n \geq 1$.*

1. Π^1_n*(L-uniformization) is the following statement:*
   For every Π^1_n subset of the plane A, if $\mu(\{x : A_x = \emptyset\}) = 0$, then there is a **Borel** *function $f : \mathbb{R} \to \mathbb{R}$ such that $\mu(\{x : f(x) \in A_x\}) = 1$.*

2. Π^1_n*(B-uniformization) is the following statement:*
   For every Π^1_n subset of the plane A, if $\{x : A_x = \emptyset\}$ is meager, then there is a **Borel** *function $f : \mathbb{R} \to \mathbb{R}$ such that $\{x : f(x) \in A_x\}$ is comeager.*

Motivated by the last definition we define the following concept:

Definition 5.2.2 *Let $n \geq 1$. $\Pi^1_n(\mathbb{P}$-uniformization) (or as we will mention it* **Souslin Uniformization***) is the following statement:*
For every Π^1_n subset of the plane A, if $\{x : A_x = \emptyset\} \in \mathcal{I}$, then there is a **Borel** *function $f : \mathbb{R} \to \mathbb{R}$ such that $\{x : f(x) \in A_x\}^c \in \mathcal{I}$.*
The invariant definition comes for $\Sigma^1_n(\mathbb{P}$-uniformization).

Corollary 5.2.3 *$\Sigma^1_n(\mathbb{P})$-uniformization iff $\Pi^1_{n-1}(\mathbb{P})$-uniformization.*

PROOF The forward direction is obvious. The backward direction is as follows: Take a Σ^1_n subset of the plane A, and assume that $\{x : A_x = \emptyset\} \in \mathcal{I}$. $A = \{(x, y) : \varphi(x, y)\}$ where φ is a Σ^1_n-formula. $\varphi(x, y) = \exists z \psi(x, y, z)$, where ψ is a Π^1_{n-1}-formula. The idea is to use the function guaranteed from the $\Pi^1_n(\mathbb{P})$-uniformization to replace the "$\exists$" sign in φ. Take

$$\phi : \mathbb{R}^2 \overset{1:1 \text{ onto}}{\longrightarrow} \mathbb{R}$$

to be a **Borel** function mapping $\mathbb{R}^2$ to $\mathbb{R}$. By our assumption on A, $E = \{x : A_x = \emptyset\} \in \mathcal{I}$. So,

$$(\forall x \notin E)(\exists y, z\ \psi(x, y, z)).$$

Therefore, we have the needed assumption for the set $B = \{\langle x, \phi(y, z)\rangle : \psi(x, y, z)\}$. But this is a Π^1_{n-1}-set, so by the uniformization for $\Pi^1_{n-1}(\mathbb{P})$ we have a function f such that

$$I = \{x : f(x) \notin B_x\} \in \mathcal{I} \tag{5.1}$$

Take $g(x, y) = x$. Then $F(x) = g(\phi^{-1}(f(x)))$ is the needed function $((\forall x \notin I)(F(x) \in A_x))$. ■

The following lemma is due to H. Woodin ([Wo] 1,2):

Lemma 5.2.4 *1. Π^1_n(L-uniformization) implies Σ^1_{n+2}-absoluteness for Random.*

2. Π^1_n(B-uniformization) implies Σ^1_{n+2}-absoluteness for Cohen.

We will now rephrase and prove that lemma for our more general case:

Lemma 5.2.5 *Let $n \geq 1$. $\Pi^1_n(\mathbb{P}$-uniformization) implies Σ^1_{n+2}-absoluteness for $\mathbb{P}$.*

PROOF Let us prove for $n = 1$. The general case follows by induction on the complexity of the formula.

Let $\exists x \forall y \varphi(x, y, z)$ be a Σ^1_3-formula with parameters in V, where φ is Σ^1_1. Suppose that v is a $\mathbb{P}$-real over V and $V[v] \models \exists x \forall y \varphi(x, y, a)$, for some $a \in \mathbb{R} \cap V$.

Let b be a witness so that $V[v] \models \forall y \varphi(b, y, a)$. Choose in V a term τ for b. τ may be chosen as a **Borel** function g such that

$$V[v] \models \forall y \varphi(g(v), y, a) \tag{5.2}$$

(see 5.1.1).

Suppose $V \models \forall x \exists y \neg\varphi(x, y, a)$. Then, $V \models \forall x \exists y \neg\varphi(g(x), y, a)$. Let $A = \{(x, y) : \neg\varphi(g(x), y, a)\}$. By $\Pi^1_1(\mathbb{P}$-uniformization), there is a **Borel** function f such that $\{x : (g(x), f(x)) \in A\}^c \in \mathcal{I}$. Choose a **Borel** set of $B \subseteq \{x : (g(x), f(x)) \in A\}$ such that $B^c \in \mathcal{I}$. Hence,

$$V \models \forall x (x \in B \Rightarrow \neg\varphi(g(x), f(x), a)).$$

Since $\neg\varphi$ is Π^1_1, $\forall x(x \in B \Rightarrow \neg\varphi(g(x), f(x), a))$ is Π^1_2 with the **Borel** codes for B, f, g as additional parameters (see 5.1.2). So,

$$V[v] \models \forall x (x \in B \Rightarrow \neg\varphi(g(x), f(x), a)).$$

But since v is a $\mathbb{P}$-real over V, and since the complement of B is a **Borel** set in the ideal $\mathcal{I}$ in V, $v \in B$ (see 3.4.3). Therefore, $V[v] \models \neg\varphi(g(v), f(v), a)$, which contradicts (5.2) above.

The other direction is simply by Shoenfield's theorem (cf [Je] Ch. 41) which gives us Σ^1_2-absoluteness, and in the induction step - by the induction hypothesis. ■

Lemma 5.2.6 *Fix $n > 0$.*
Assume Π^1_{n+1}-absoluteness for $\mathbb{P}$. Take $\varphi \in (\Sigma^1_n \cup \Pi^1_n)$ and τ the canonical $\mathbb{P}$-name for a $\mathbb{P}$-real. Assign $p = \{x : \varphi(x)\}$. Then

1. *if $p/\mathcal{I} \in \mathbb{P}$ then*
$$[\![\varphi(\tau)]\!] \;=\; p/\mathcal{I}.$$
2. *if p contains a subset q such that $q/\mathcal{I} \in \mathbb{P}$, then*
$$[\![\varphi(\tau)]\!] \geq q/\mathcal{I}.$$

Proof Let us prove *2* first, and then *1* will follow easily. Take p and q as mentioned above. Take a **Borel** set $F \subseteq q$ such that $F/\mathcal{I} = q/\mathcal{I}$ (By the assumption on q, where I is the σ-ideal mentioned above). Take $r = F/\mathcal{I}$. I claim that $r \Vdash \varphi(\tau)$. Take $a \in F^*$ $\mathbb{P}$-real over V (If there is none, then $q \in \mathcal{I}$ and we are done). Then
$$V[a] \models \varphi(a),$$
since by $\Pi^1_{n+1}[\Pi^1_n]$-absoluteness for $\mathbb{P}$,
$$V \models \forall x \in F(\varphi(x)) \Rightarrow V^{\mathbb{B}} \models \forall x \in F(\varphi(x)).$$
But $V[a] \models \tau[a] = a$ (τ is the canonical $\mathbb{P}$-name for a $\mathbb{P}$-real). So $V[a] \models \varphi(\tau)$ (and this is for each $a \in F^*$ $\mathbb{P}$-real over V), thus $r \Vdash \varphi(\tau)$ and
$$[\![\varphi(\tau)]\!] \geq q/\mathcal{I}$$
■

Now, for the first part of the lemma, notice that p satisfies the assumptions given in the second part, of q. Thus it is obvious that
$$[\![\varphi(\tau)]\!] \geq p/\mathcal{I}$$
For the the equality case, just observe that for '$\leq$' we have
$$[\![\neg\varphi(\tau)]\!] \geq \{x : \neg\varphi(x)\}/\mathcal{I}$$
(By the assumption of Π^1_{n+1}-absoluteness), which implies
$$[\![\varphi(\tau)]\!] \leq p/\mathcal{I}$$

Thus
$$[\![\varphi(\tau)]\!] = p/\mathcal{I}$$

■

Fact 5.2.7 *Let $A \subseteq \mathbb{R}$. $(A \cap [\![\varphi(\tau)]\!]) \notin \mathcal{I} \;\Rightarrow\; [\![\varphi(\tau)]\!] \notin \mathcal{I}$. (This is especially correct for the case of A as our p from the last lemma)*

PROOF $[\![\varphi(\tau)]\!] \in \mathcal{I} \;\Rightarrow\; A \cap [\![\varphi(\tau)]\!] \in \mathcal{I}$ ($\mathcal{I}$ is an ideal). ■

As an example of the last lemma's use - one can observe the following corollary:

Corollary 5.2.8 *Fix $n > 0$.*

1. *Assume $\Sigma^1_n(L)$. Assume also that Π^1_{n+1}-absoluteness for Random holds. Take $\varphi \in (\Sigma^1_n \cup \Pi^1_n)$ and τ the canonical random name for a random real. Then*
$$\mu([\![\varphi(\tau)]\!]) \;=\; \mu(\{x : \varphi(x)\})$$
2. *Assume $\Sigma^1_n(B)$. Assume further that Π^1_{n+1}-absoluteness for Cohen holds. Take $\varphi \in (\Sigma^1_n \cup \Pi^1_n)$ and τ the canonical cohen name for a cohen real. Then*
$$\{x : \varphi(x)\} \textit{is not meager} \iff [\![\varphi(\tau)]\!] \textit{is not meager}$$

■

Corollary 5.2.9 *Let $\mathbb{P}$ be a countably generated forcing notion. Fix $n > 0$. Assume Π^1_{n+1}-absoluteness for $\mathbb{P}$. Take $\varphi \in (\Sigma^1_n \cup \Pi^1_n)$. Assign $p = \{x : \varphi(x)\}$, and ψ the dense embedding mentioned in theorem 3.2.13. Then there is a $\mathbb{P}$-name τ for a $\mathbb{P}$-real such that:*

1. *if $p/\mathcal{I}_{\mathbb{P}} \in \psi(\mathbb{P})$ then*
$$\psi([\![\varphi(\tau)]\!]) \;=\; p/\mathcal{I}_{\mathbb{P}}$$
2. *if p contains a subset q such that $q/\mathcal{I}_{\mathbb{P}} \in \psi(\mathbb{P})$, then*
$$\psi([\![\varphi(\tau)]\!]) \geq q/\mathcal{I}_{\mathbb{P}}$$

■

We will now use the notion defined in 3.4.1 to prove the opposite direction to 5.2.5 under the assumptions of this notion.

Lemma 5.2.10 *Assume $\Sigma^1_n(\mathcal{I})$. Then, $\Sigma^1_{n+2}(\mathbb{P})$-absoluteness implies $\Pi^1_n(\mathbb{P})$-uniformization.*

PROOF Let $A = \{(x, y) : \varphi(x, y)\}$ be a Π^1_n subset of the plane. Suppose that $\{x : A_x = \emptyset\} \in \mathcal{I}$. Take C a **Borel** subset of $\mathcal{I}$, with $\{x : A_x = \emptyset\} \subseteq C$. Let $B = \{(x, y) : x \in C\}$. Thus B is a **Borel** set in $\mathcal{I} \times \mathcal{P}(\mathbb{R})$. Let $\psi(x, y)$ be an arithmetical formula that defines B. Then

$$V \models \forall x(\exists y\varphi(x, y) \vee \exists y\psi(x, y))$$

By Σ^1_{n+2}-absoluteness for $\mathbb{P}$

$$V^{\mathbb{P}} \models \forall x(\exists y\varphi(x, y) \vee \exists y\psi(x, y))$$

Let τ be the canonical name for a $\mathbb{P}$-real in V.

$$V^{\mathbb{P}} \models \exists y\varphi(\tau, y) \vee \exists y\psi(\tau, y)$$

Moreover, if a is a $\mathbb{P}$-real over V, then $V[a] \models \tau[a] = a$. But since $\{x : B_x \neq \emptyset\}$ is a **Borel** set contained in $\mathcal{I}$ in V, $a \notin \{x : B_x \neq \emptyset\}^*$. Hence

$$V^{\mathbb{P}} \models \exists y\varphi(\tau, y)$$

Let σ be a $\mathbb{P}$-name for a real such that

$$V^{\mathbb{P}} \models \varphi(\tau, \sigma)$$

Then we can find a Borel function f such that for each $\mathbb{P}$-real a, $V[a] \models \sigma[a] = f(a)$. So

$$V^{\mathbb{P}} \models \varphi(\tau, f(\tau)) \tag{5.3}$$

Now Assume $\{x : \neg\varphi(x, f(x))\} \notin \mathcal{I}$. Take $p = [\![\neg\varphi(\tau, f(\tau))]\!]$. Take $a \in p^*$ a $\mathbb{P}$-real over V (one may do so, since by corollary 5.2.6, $p \notin \mathcal{I}$). Then, $V[a] \models \neg\varphi(a, f(a))$ (p forces that), but that contradicts (5.3) above.

Therefore $\{x : \neg\varphi(x, f(x))\} \in \mathcal{I}$. ■

For our theorem to be complete, we need another property which is shown in the following lemma:

Lemma 5.2.11 *Π^1_n($\mathbb{P}$-uniformization) implies $\Sigma^1_n(\mathcal{I})$.*

PROOF Take a set $C \in \Sigma^1_n$. Take $A = C^c$. Take $A' = A \times \{0\} \cup \{\langle x, x\rangle : x \in A^c\}$. By uniformization, we have a Borel function f such that

$$\mu(\{x : f(x) = x \vee f(x) = 0\}) = 1.$$

Take $B = \{x : f(x) = 0\}$. Then B is Borel and $\mu(A \triangle B) = 0$. Thus A has the $\mathbb{P}.p$, and therefore C has the $\mathbb{P}.p$. ■

Theorem 5.2.12 *$\Sigma^1_{n+2}(\mathbb{P})$-absoluteness + $\Sigma^1_n(\mathcal{I}) \iff \Pi^1_n(\mathbb{P})$-uniformization.*

PROOF Obvious from lemmas 5.2.10, 5.2.5 and 5.2.11. ■

To emphasis the use of the previous theorem, let me give an example of the use for Random and Cohen forcing notions.

Corollary 5.2.13

1. *Σ^1_{n+2}-absoluteness for Random + $\Sigma^1_n(L)$ iff $\Pi^1_n(L)$-uniformization.*
2. *Σ^1_{n+2}-absoluteness for Cohen + $\Sigma^1_n(B)$ iff $\Pi^1_n(B)$-uniformization.*

■

Chapter 6

Souslin Absoluteness *vs* Projective regularity

In our last chapter we will use technical tools about Souslin forcing and the theorems proved in the previouse chapter, to show that Souslin absoluteness implies $\Delta^1_4(L)$ and $\Delta^1_4(B)$.

6.1 Latest Results

We have several goals in our research in descriptive set theory:

1. To find a statement about the reals that explains completely the theory of the reals in Solovay models.
2. To find a combinatorial statement equivalent to "Projective measurability" (as well as the Baire Property).

We have seen some result concerning the first direction in the previous chapters. In the second direction we have the following results:

Theorem 6.1.1 ([Ju2]) *1. Δ^1_2-measurability iff Σ^1_3 (Random)-Absolute*

2. Δ^1_2-categoricity iff Σ^1_3 (Cohen)-Absolute

3. Σ^1_2-measurability iff Σ^1_3 (Amoeba)-Absolute

4. Σ^1_2-categoricity iff Σ^1_3 (Hechler)-Absolute

Theorem 6.1.2 ([Ju2]) *1. Σ^1_4(Random)-Absolute + Σ^1_3(Amoeba)-Absolute $\rightarrow \Delta^1_3$-measurability*

2. Σ^1_4 (Cohen)-Absolute + Σ^1_3 (Hechler)-Absolute $\rightarrow \Delta^1_3$-categoricity

Shelah proved the following :

Theorem 6.1.3 (cf **[Ju2] p.8**) $\Sigma^1_3(L) \Rightarrow (\forall r \in \mathbb{R})(\omega_1^{\mathbb{L}[r]} < \omega_1)$.

Recently J. Brendle, using the ideas of [Ju2], proved the following

Theorem 6.1.4 (cf **[Ju2] p.14**) *Σ^1_4 (Amoeba)-Absolute $\rightarrow \Sigma^1_3$-measurability.*

Corollary 6.1.5 (cf **[Ju2] p.14**) *Σ^1_4 (Amoeba)-Absolute $\rightarrow \Sigma^1_3$-categoricity.*

6.2 Souslin Absoluteness & Δ^1_4 Measurability

In this section we will use the uniformization properties mentioned in the previouse chapter to show Δ^1_4-measurability.

Suppose τ is the canonical Random name for a random real.

Lemma 6.2.1 *Assume That for some $t \in \mathbb{R}$, $\mathbb{L}[t]^{\mathbb{B}} \models \varphi(\tau) \iff \psi(\tau)$. Then:*

1. $\mathbb{L}[t] \models [\![\varphi(\tau)]\!] \cap [\![\psi(\tau)]\!] = 0$
2. $\mathbb{L}[t] \models [\![\varphi(\tau)]\!] \cup [\![\psi(\tau)]\!] = 1$

Proof

1. Suppose otherwise. So, there is a Borel set p of positive measure such that $[\![\varphi(\tau)]\!] \cap [\![\psi(\tau)]\!] = p$. But then, if $r \in p$ is random over $\mathbb{L}[t]$,
$$\mathbb{L}[t][r] \models \varphi(\tau[r]) \wedge \psi(\tau[r]).$$
2. Suppose otherwise. So, there is a Borel set q of positive measure such that $[\![\varphi(\tau)]\!]^c \cap [\![\psi(\tau)]\!]^c = q$. But then, if $r \in q$ is random over $\mathbb{L}[t]$,
$$\mathbb{L}[t][r] \models \neg\varphi(\tau[r]) \wedge \neg\psi(\tau[r]).$$

■

Theorem 6.2.2 *Σ^1_4-absoluteness for Amoeba + Σ^1_5-absoluteness for Random implies $\Delta^1_4(L)$.*

Proof Let $A = \{x : \varphi(x)\}$, $B = \{x : \psi(x)\}$, where φ, ψ are Σ^1_4-formulas with parameters in V. Suppose

$$V \models \forall x(\varphi(x) \Leftrightarrow \neg\psi(x)).$$

i.e., A is a Δ^1_4 set of reals in V. Then, by Σ^1_5-absoluteness for Random

$$V^{\mathbb{B}} \models \forall x(\varphi(x) \Leftrightarrow \neg\psi(x)).$$

Now, suppose τ is the canonical Random name for a random real.

Claim 6.2.3 *There exists $t \in \mathbb{R}$ such that $\mu(A) = \mu([\![\varphi(\tau)]\!]_{\mathbb{B}^{\mathbb{L}[t]}})$.*

Proof of claim: Let φ_1, ψ_1 be Π^1_3-formulas such that $\varphi(x) = \exists y \varphi_1(x, y)$, $\psi(x) = \exists y \psi_1(x, y)$.

$$V \models \forall x(\varphi(x) \vee \psi(x)) \Longrightarrow V \models \forall x(\exists y(\varphi_1(x, y)) \vee \exists y(\psi_1(x, y)))$$

Thus, by (6.1.4)

$$\Sigma^1_4(\mathbb{A})\text{-absoluteness} \Longrightarrow V \models \Sigma^1_3(L).$$

Also since in V, $\Sigma^1_5(\mathbb{B})$-absoluteness holds, we have by (5.2.13) that $\Pi^1_3(L)$-uniformization holds. Thus there is a borel function f, and a null set N such that

$$V \models \forall x(x \notin N \Rightarrow \varphi_1(x, f(x)) \vee \psi_1(x, f(x))).$$

Let t code the parameters of φ, ψ and f. By $\Sigma^1_3(L)$ (actually we need only $\Sigma^1_2(L)$), we have that $\mu(Ra(\mathbb{L}[t])) = 1$. Take $r \in V$ Random over $\mathbb{L}[t]$. Also assume $r \notin N$. Then

$$V \models \varphi_1(r, f(r)) \vee \psi_1(r, f(r)).$$

Assume w.l.o.g that $V \models \varphi_1(r, f(r))$. By (5.1.2) "$\varphi_1(r, f(r))$" is a Π^1_3-formula in the additional parameters, borel codes for f, r. Thus by Shoenfield's Absoluteness Theorem (downward absoluteness) $\mathbb{L}[t][r] \models \varphi_1(r, f(r))$. Thus

$$\mathbb{L}[t][r] \models \varphi(r).$$

Therefore we get that $r \in [\![\varphi(\tau)]\!]_{\mathbb{B}^{\mathbb{L}[t]}}$. This is true for each $r \in A \setminus N$. Thus

$$\mu(A \setminus [\![\varphi(\tau)]\!]_{\mathbb{B}^{\mathbb{L}[t]}}) = 0.$$

The same process shows that $\mu(B \setminus [\![\psi(\tau)]\!]_{\mathbb{B}^{\mathbb{L}[t]}}) = 0$. Now we only need to show that the assumptions of lemma (6.2.1), hold for t. But, we have just seen that for each $r \in V$ Random over $\mathbb{L}[t]$,

$$V \models \varphi(r) \Rightarrow \mathbb{L}[t][r] \models \varphi(r),$$

and the same goes for ψ. So we get that

$$\mathbb{L}[t][r] \models \neg\psi(r) \Rightarrow V \models \neg\psi(r) \iff V \models \varphi(r) \Rightarrow \mathbb{L}[t][r] \models \varphi(r).$$

The same goes for the other direction. Thus the assumptions of lemma (6.2.1), hold. Thus

$$\mu([\![\varphi(\tau)]\!]) = \mu(A).$$

■

6.3 Δ^1_4-Categoricity and the general case

We will now establish the general connection which was used in the previouse section to show $\Delta^1_4(L)$, to show $\Delta^1_4(\mathcal{I})$ given $\Sigma^1_3(\mathcal{I})$. For this we need a powerfull assumption. We assume that $\forall t \in \mathbb{R}$

$$V \models (Pr(\mathbb{L}[t]))^c \in \mathcal{I}.$$

The question, of the validity of this assumption under $\Sigma^1_3(\mathcal{I})$ is still open. We know though, that it is true under $\Sigma^1_3(L)$.

Suppose τ is the canonical $\mathbb{P}$-name for a $\mathbb{P}$-real.

Lemma 6.3.1 *Assume $\mathcal{M}$ is a transitive model of ZFC. Assume further that $\mathcal{M}^{\mathbb{B}} \models \varphi(\tau) \iff \psi(\tau)$. Then:*

1. $\mathcal{M} \models [\![\varphi(\tau)]\!] \cap [\![\psi(\tau)]\!] = 0$
2. $\mathcal{M} \models [\![\varphi(\tau)]\!] \cup [\![\psi(\tau)]\!] = 1$

Proof

1. Suppose otherwise. So, there is a Borel set $p \notin \mathcal{I}$ such that $[\![\varphi(\tau)]\!] \cap [\![\psi(\tau)]\!] = p$. But then, if $r \in p$ is a $\mathbb{P}$-real over $\mathcal{M}$,

$$\mathcal{M} \models \varphi(\tau[r]) \wedge \psi(\tau[r]).$$

2. Suppose otherwise. So, there is a Borel set $q \notin \mathcal{I}$ such that $[\![\varphi(\tau)]\!]^c \cap [\![\psi(\tau)]\!]^c = q$. But then, if $r \in q$ is a $\mathbb{P}$-real over $\mathcal{M}$,

$$\mathcal{M} \models \neg\varphi(\tau[r]) \wedge \neg\psi(\tau[r]).$$

■

Theorem 6.3.2 *Assume $\forall t \in \mathbb{R}$, $V \models (Pr(\mathbb{L}[t]))^c \in \mathcal{I}$. Then $\Sigma^1_3(\mathcal{I})$ + $\Sigma^1_5(\mathbb{P})$-absoluteness implies $\Delta^1_4(\mathcal{I})$.*

Proof Let $A = \{x : \varphi(x)\}$, $B = \{x : \psi(x)\}$, where φ, ψ are Σ^1_4-formulas with parameters in V. Suppose

$$V \models \forall x(\varphi(x) \Leftrightarrow \neg\psi(x)).$$

i.e., A is a Δ^1_4 set of reals in V. Then, by $\Sigma^1_5(\mathcal{I})$-absoluteness

$$V^{\mathbb{P}} \models \forall x(\varphi(x) \Leftrightarrow \neg\psi(x)).$$

Now, suppose τ is the canonical $\mathbb{P}$-name for a $\mathbb{P}$-real.

Claim 6.3.3 *There exists $t \in \mathbb{R}$ such that $A \triangle [\![\varphi(\tau)]\!]_{\mathbb{B}^{\mathbb{L}[t]}} \in \mathcal{I}$.*

PROOF of claim: Let φ_1, ψ_1 be Π^1_3-formulas such that $\varphi(x) = \exists y \varphi_1(x, y)$, $\psi(x) = \exists y \psi_1(x, y)$.

$$V \models \forall x(\varphi(x) \vee \psi(x)) \Longrightarrow V \models \forall x(\exists y(\varphi_1(x, y)) \vee \exists y(\psi_1(x, y)))$$

Since $V \models \Sigma^1_3(\mathcal{I})$ and in V, $\Sigma^1_5(\mathbb{P})$-absoluteness holds, we have by (5.2.12) that $\Pi^1_3(\mathcal{I})$-uniformization holds. Thus there is a borel function f, and a set $I \in \mathcal{I}$ such that

$$V \models \forall x(x \notin I \Rightarrow \varphi_1(x, f(x)) \vee \psi_1(x, f(x))).$$

Let t code the parameters of φ, ψ and f. By our assumption $V \models (Pr(\mathbb{L}[t]))^c \in \mathcal{I}$. Take $r \in V$ a $\mathbb{P}$-real over $\mathbb{L}[t]$. Also assume $r \notin I$. Then

$$V \models \varphi_1(r, f(r)) \vee \psi_1(r, f(r)).$$

Assume w.l.o.g that $V \models \varphi_1(r, f(r))$. By (5.1.2) "$\varphi_1(r, f(r))$" is a Π^1_3-formula in the additional parameters, borel codes for f, r. Thus by Shoenfield's Absoluteness Theorem (downward absoluteness) $\mathbb{L}[t][r] \models \varphi_1(r, f(r))$. Thus

$$\mathbb{L}[t][r] \models \varphi(r).$$

Therefore we get that $r \in [\![\varphi(\tau)]\!]_{\mathbb{B}^{\mathbb{L}[t]}}$. This is true for each $r \in A \setminus I$. Thus

$$A \setminus [\![\varphi(\tau)]\!] \in \mathcal{I}.$$

The same process shows that $B \setminus [\![\psi(\tau)]\!] \in \mathcal{I}$. Now we only need to show that the assumptions of lemma (6.3.1), hold for t. But, we have just seen that for each $r \in V$, $\mathbb{P}$-real over $\mathbb{L}[t]$,

$$V \models \varphi(r) \Rightarrow \mathbb{L}[t][r] \models \varphi(r),$$

and the same goes for ψ. So we get that

$$\mathbb{L}[t][r] \models \neg\psi(r) \Rightarrow V \models \neg\psi(r) \iff V \models \varphi(r) \Rightarrow \mathbb{L}[t][r] \models \varphi(r).$$

The same goes for the other direction. Thus the assumptions of lemma (6.3.1), hold. Thus

$$[\![\varphi(\tau)]\!] \triangle A \in \mathcal{I}.$$

■

Lemma 6.3.4 $\Sigma^1_3(L) \Longrightarrow (\forall t \in \mathbb{R})(V \models (Pr(\mathbb{L}[t]))^c \in \mathcal{I})$.

PROOF By (6.1.3) we know that $\Sigma^1_3(L) \Rightarrow (\forall r \in \mathbb{R})(\omega_1^{\mathbb{L}[r]} < \omega_1)$. So $\forall t \in \mathbb{R}^V (\mathbb{R}^{\mathbb{L}[t]}$ is countable). Also by (3.4.3) we know that a real number is a $\mathbb{P}$-real over $\mathbb{L}[t]$ iff it does not belong to any Borel set $I \in \mathcal{I}$ with a code in $\mathbb{L}[t]$. Thus

$$x \in Pr(\mathbb{L}[t]) \iff x \notin \bigcup \{X : X \in \mathcal{I} \cap \mathbb{L}[t]\} \in \mathcal{I}^V.$$

Therefore

$$(\forall t \in \mathbb{R})(V \models (Pr(\mathbb{L}[t]))^c \in \mathcal{I}).$$

■

Corollary 6.3.5 *$\Sigma^1_4(\mathbb{A})$-absoluteness + $\Sigma^1_3(\mathcal{I})$ + $\Sigma^1_5(\mathbb{P})$-absoluteness implies $\Delta^1_4(\mathcal{I})$.*

PROOF using the previouse theorem and lemma. ■

Corollary 6.3.6 *Souslin-Absoluteness implies $\Delta^1_4(\mathcal{I})$ (and therefore $\Delta^1_4(B)$).*
■

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