The Mutilated Checkerboard in Set Theory

John McCarthy
Computer Science Department
Stanford University
jmc@cs.stanford.edu
http://www-formal.stanford.edu/jmc/

April 27, 2001

An 8 by 8 checkerboard with two diagonally opposite squares removed cannot be covered by dominoes each of which covers two rectilinearly adjacent squares. We present a set theory description of the proposition and an informal proof that the covering is impossible. While no present system that I know of will accept either the formal description or the proof, I claim that both should be admitted in any *heavy duty set theory.*[1]

We have the definitions

$$Board = Z8 \times Z8, \tag{1}$$

$$mutilated\text{-}board = Board - \{(0,0),(7,7)\}, \tag{2}$$

$$\begin{aligned} &domino\text{-}on\text{-}board(x) \equiv (x \subset Board) \wedge card(x) = 2 \\ &\wedge (\forall x1\ x2)(x = \{x1, x2\} \rightarrow adjacent(x1, x2)) \end{aligned} \tag{3}$$

and

$$\begin{aligned} &adjacent(x1, x2) \equiv |c(x1,1) - c(x2,1)| = 1 \\ &\wedge c(x1,2) = c(x2,2) \\ &\vee |c(x1,2) - c(x2,2)| = 1 \wedge c(x1,1) = c(x2,1). \end{aligned} \tag{4}$$

If we are willing to be slightly tricky, we can write more compactly

$$adjacent(x1, x2) \equiv |c(x1,1) - c(x2,1)| + |c(x1,2) - c(x2,2)| = 1, \tag{5}$$

but then the proof might not be so obvious to the program.

Next we have.

[1] The Mizar proof checker accepts the definitions essentially as they are, but the first proof in Mizar is 400 lines.

$$\begin{aligned}&partial\text{-}covering(z)\\&\equiv (\forall x)(x \in z \rightarrow domino\text{-}on\text{-}board(x))\\&\wedge(\forall x\ y)(x \in z \wedge y \in z \rightarrow x = y \vee x \cap y = \{\})\end{aligned} \tag{6}$$

Theorem:

$$\neg(\exists z)(partial\text{-}covering(z) \wedge \bigcup z = mutilated\text{-}board) \tag{7}$$

Proof:

We define

$$x \in Board \rightarrow color(x) = rem(c(x,1) + c(x,2), 2) \tag{8}$$

$$\begin{aligned}&domino\text{-}on\text{-}board(x) \rightarrow\\&(\exists u\ v)(u \in x \wedge v \in x \wedge color(u) = 0 \wedge color(v) = 1),\end{aligned} \tag{9}$$

$$\begin{aligned}&partial\text{-}covering(z) \rightarrow\\&card(\{u \in \bigcup z | color(u) = 0\})\\&= card(\{u \in \bigcup z | color(u) = 1\}),\end{aligned} \tag{10}$$

$$\begin{aligned}&card(\{u \in mutilated\text{-}board | color(u) = 0\})\\&\neq card(\{u \in mutilated\text{-}board | color(u) = 1\}),\end{aligned} \tag{11}$$

and finally

$$\neg(\exists z)(partial\text{-}covering(z) \quad \wedge \quad mutilated\text{-}board = \bigcup z) \tag{12}$$

Q.E.D.