# CS 323, Fall 2000
# Problem Set #2 Solutions: Circumscription
Handout #9

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Remember!

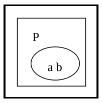
$$\begin{aligned} Circ[A(P);P] &\equiv [A(P) \wedge \forall \phi.(A(\phi) \wedge \phi \leq P) \rightarrow \phi = P] \\ Circ[A(P,Z);P;Z] &\equiv [A(P,Z) \wedge \forall \phi.(\exists \zeta.A(\phi,\zeta) \wedge \phi \leq P) \rightarrow \phi = P] \end{aligned} \tag{1}$$

## 1 Problem Set #1 Solutions, section 1.3

1. [**5 points**] $A(P) \equiv [P(a) \wedge P(b)]$

   **Solution:**

$$Circ[A(P);P] \equiv [\forall x.P(x) \equiv (x = a) \vee (x = b)] \tag{2}$$




Proof:

$\rightarrow$:

| | |
|---|---|
| $P(a)$ | 1. Given from Circ[*A(P)*; P] |
| $P(b)$ | 2. Given from Circ[*A(P)*; P] |
| $\forall \phi.[(\phi(a) \wedge \phi(b) \wedge \forall x.\phi(x) \rightarrow P(x)) \rightarrow$ $(\forall x.\phi(x) \equiv P(x))]$ | 3. Given from: Circ[*A(P)*; P] |
| $((a = a \vee a = b) \wedge (b = a \vee b = b)$ $\wedge \forall x.(x = a) \vee (x = b) \rightarrow P(x)) \rightarrow$ $(\forall x.(x = a) \vee (x = b) \equiv P(x))$ | 4. Substitute $x = a \vee x = b$ for $\phi(x)$ (Universal instantiation in (3).) |
| $\forall x.(x = a) \vee (x = b) \equiv P(x)$ | 5. Modus Ponens (1, 3) ($a = a$ is true, and $b = b$ is true, and $P(a) \wedge P(b)$ is equivalent to $\forall x.(x = a) \vee (x = b) \rightarrow P(x)$) |

(3)

We have to prove both $P(a) \wedge P(b)$ and $\forall \phi.[(\phi(a) \wedge \forall x.(\phi(x) \rightarrow P(x))) \rightarrow (\forall x.\phi(x) \equiv P(x))]$ from $\forall x.(x = a) \vee (x = b) \equiv P(x)$. The first is trivial (just substitute into the universal), so let's try the second. Let's assume the antecedent, for arbitrary $\phi$:

$\leftarrow$:

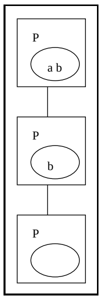
| | |
|---|---|
| $\forall x.(x = a) \vee (x = b) \equiv P(x)$ | 1. Given |
| $\phi(a) \wedge \phi(b) \wedge \forall x.\phi(x) \rightarrow P(x)$ | 2. Assumed |
| $\phi(a) \wedge \phi(b) \wedge \forall x.\phi(x) \rightarrow [(x = a) \vee (x = b)]$ | 3. Substitute for def. of $P(x)$ |
| $\phi(a) \wedge \phi(b)$ | 4. $\wedge$-elimination of (3) |
| $\forall x.\phi(x) \rightarrow [(x = a) \vee (x = b)]$ | 5. $\wedge$-elimination of (3) |
| $\forall x.(x = a) \vee (x = b) \rightarrow \phi(x)$ | 6. Equivalent to (4) |
| $\forall x.(x = a) \vee (x = b) \equiv \phi(x)$ | 7. Def. of $\equiv$ (5, 6) |
| $\forall x.\phi(x) \equiv P(x)$ | 8. Transitivity of $\equiv$ (1, 7) |

(4)

2. [**5 points**] $A(P) \equiv [P(a) \rightarrow P(b)]$

**Solution:**

$$Circ[A(P); P] \equiv [\forall x. P(x) \equiv \bot] \tag{5}$$




Proof:

$\rightarrow$:

| | |
|---|---|
| $P(a) \rightarrow P(b)$ | 1. Given from Circ[$A(P)$; P] |
| $\forall\phi.[(([\phi(a) \rightarrow \phi(b)] \wedge \forall x.\phi(x) \rightarrow P(x)) \rightarrow$ $(\forall x.\phi(x) \equiv P(x))]$ | 2. Given from: Circ[$A(P)$; P] |
| $((\bot\rightarrow\bot) \wedge \forall x.\ \bot\rightarrow P(x)) \rightarrow (\forall x.\ \bot\equiv P(x))$ | 3. Substitute $\bot$ for $\phi(x)$ (Universal instantiation in (2).) |
| $\forall x.\ \bot\equiv P(x)$ | 4. Modus Ponens (1, 3) ($\bot\rightarrow\bot$ holds, and $\forall x.\ \bot\rightarrow P(x)$ is true.) |

(6)

We have to prove both $P(a) \rightarrow P(b)$ and $\forall\phi.[((\phi(a) \rightarrow \phi(b)) \wedge \forall x.(\phi(x) \rightarrow P(x))) \rightarrow (\forall x.\phi(x) \equiv P(x))]$ from $\forall x.\ \bot \equiv P(x)$. The first is trivial ($\bot \rightarrow \bot$), so let's try the second. Let's assume the antecedent, for arbitrary $\phi$:

$\leftarrow$:

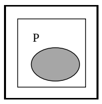
| | |
|---|---|
| $\forall x.\ \bot\equiv P(x)$ | 1. Given |
| $(\phi(a)\rightarrow\phi(b)) \wedge \forall x.\phi(x) \rightarrow P(x)$ | 2. Assumed |
| $(\phi(a)\rightarrow\phi(b)) \wedge \forall x.\phi(x) \rightarrow\bot$ | 3. Substitute for def. of $P(x)$ |
| $\phi(a)\rightarrow\phi(b)$ | 4. $\wedge$-elimination of (3) |
| $\forall x.\phi(x) \rightarrow\bot$ | 5. $\wedge$-elimination of (3) |
| $\forall x.\ \bot \rightarrow\phi(x)$ | 6. Tautology |
| $\forall x.\ \bot\equiv \phi(x)$ | 7. Def. of $\equiv$ (5, 6) |
| $\forall x.\phi(x) \equiv P(x)$ | 8. Transitivity of $\equiv$ (1, 7) |

(7)

3. [**5 points**] $A(P) \equiv [\forall x.P(x)]$

**Solution:**

$$Circ[A(P); P] \equiv [\forall x.P(x)] \equiv [\forall x.P(x) \equiv true] \tag{8}$$




Proof:

$\rightarrow$:

$\forall x.P(x)$ 1. Given from Circ[$A(P)$; P] (9)

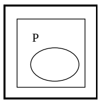
We have to prove both $\forall x.P(x)$ and $\forall\phi.[(\forall x.\phi(x) \wedge \forall x.(\phi(x) \rightarrow P(x))) \rightarrow (\forall x.\phi(x) \equiv P(x))]$ from $\forall x.P(x) \equiv true$. The first is trivial, so let's try the second. Let's assume the antecedent, for arbitrary $\phi$:

$$\begin{array}{ll} \leftarrow: & \\ \forall x.true \equiv P(x) & \text{1. Given} \\ \forall x.\phi(x) \wedge \forall x.\phi(x) \rightarrow P(x) & \text{2. Assumed} \\ \forall x.\phi(x) \wedge \forall x.\phi(x) \rightarrow true & \text{3. Substitute for def. of } P(x) \\ \forall x.\phi(x) & \text{4. } \wedge\text{-elimination of (3)} \\ \forall x.true \equiv \phi(x) & \text{5. Reduction of (4)} \\ \forall x.\phi(x) \equiv P(x) & \text{6. Transitivity of } \equiv \text{ (1, 5)} \end{array} \tag{10}$$

4. [**5 points**] $A(P) \equiv [\forall x.\neg P(x)]$

**Solution:**

$$Circ[A(P);P] \equiv [\forall x.P(x) \equiv \perp] \equiv [\forall x.\neg P(x)] \tag{11}$$




Proof:

$$\begin{array}{ll} \rightarrow: & \\ \forall x.P(x) \equiv \perp & \text{1. Given from Circ[}A(P)\text{; P]} \end{array} \tag{12}$$

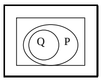
We have to prove both $\forall x.\neg P(x)$ and $\forall\phi.[(\forall x.\neg\phi(x) \wedge \forall x.(\phi(x) \rightarrow P(x))) \rightarrow (\forall x.\phi(x) \equiv P(x))]$ from $\forall x.P(x) \equiv \perp$. The first is by definition, so let's try the second. Let's assume the antecedent, for arbitrary $\phi$:

$$\begin{array}{ll} \leftarrow: & \\ \forall x.P(x) \equiv \perp & \text{1. Given} \\ \forall x.\neg\phi(x) \wedge \forall x.\phi(x) \rightarrow P(x) & \text{2. Assumed} \\ \forall x.\neg\phi(x) \wedge \forall x.\phi(x) \rightarrow \perp & \text{3. Substitute for def. of } P(x) \\ \forall x.\neg\phi(x) & \text{4. } \wedge\text{-elimination of (3)} \\ \forall x. \perp \equiv \phi(x) & \text{5. Reduction of (4)} \\ \forall x.\phi(x) \equiv P(x) & \text{6. Transitivity of } \equiv \text{ (1, 5)} \end{array} \tag{13}$$

5. [**5 points**] $A(P) \equiv [\forall x.Q(x) \rightarrow P(x)]$

**Solution:**

$$Circ[A(P);P] \equiv [\forall x.P(x) \equiv Q(x)] \tag{14}$$



Proof:

$$\begin{array}{ll}
\rightarrow: & \\
\forall x.Q(x) \rightarrow P(x) & \text{1. Given from Circ}[A(P);\text{ P}] \\
\forall \phi.[(\forall x.Q(x) \rightarrow \phi(x) \wedge \forall x.\phi(x) \rightarrow P(x)) \rightarrow & \\
\quad (\forall x.\phi(x) \equiv P(x))] & \text{2. Given from: Circ}[A(P);\text{ P}] \\
(\forall x.Q(x) \rightarrow Q(x) \wedge \forall x.Q(x) \rightarrow P(x)) \rightarrow & \\
\quad (\forall x.Q(x) \equiv P(x)) & \text{3. Substitute } Q(x) \text{ for } \phi(x) \\
 & \text{(Universal instantiation in (2).)} \\
\forall x.Q(x) \equiv P(x) & \text{4. Modus Ponens (1, 3)}
\end{array} \tag{15}$$

We have to prove both $\forall x.Q(x) \rightarrow P(x)$ and $\forall \phi.[(\forall x.Q(x) \rightarrow \phi(x) \wedge \forall x.(\phi(x) \rightarrow P(x))) \rightarrow (\forall x.\phi(x) \equiv P(x))]$ from $\forall x.Q(x) \equiv P(x)$. The first is just half of the equivalence, so let's try the second. Let's assume the antecedent, for arbitrary $\phi$:

$$\begin{array}{ll}
\leftarrow: & \\
\forall x.Q(x) \equiv P(x) & \text{1. Given} \\
\forall x.Q(x) \rightarrow \phi(x) \wedge \forall x.\phi(x) \rightarrow P(x) & \text{2. Assumed} \\
\forall x.Q(x) \rightarrow \phi(x) \wedge \forall x.\phi(x) \rightarrow Q(x) & \text{3. Substitute for def. of } P(x) \\
\forall x.\phi(x) \equiv Q(x) & \text{4. Definition of } \equiv \\
\forall x.\phi(x) \equiv P(x) & \text{5. Transitivity of } \equiv \text{ (1, 4)}
\end{array} \tag{16}$$

6. **[5 points]** $A(P) \equiv [(\forall x.Q(x) \rightarrow P(x)) \wedge P(a)]$

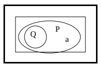
**Solution:**

**Point Breakdown**:

(a) (-2) : Wrong answer like $\forall x.(Q(x) \rightarrow P(x)) \wedge (P(x) \equiv x = a)$,

(b) (-1) : or $\forall x.P(x) \equiv x = a \wedge \forall x.P(x) \rightarrow Q(x)$, but at least a good explanation.

$$Circ[A(P); P] \equiv [\forall x.P(x) \equiv (x = a) \vee Q(x)] \tag{17}$$



Proof:

$$\begin{array}{ll}
\rightarrow: & \\
\forall x.(Q(x) \vee x = a) \rightarrow P(x) & \text{1. Given from Circ}[A(P);\text{ P}] \\
\forall \phi.[(\forall x.(Q(x) \vee x = a) \rightarrow \phi(x) \wedge \forall x.\phi(x) \rightarrow P(x)) \rightarrow & \\
\quad (\forall x.\phi(x) \equiv P(x))] & \text{2. Given from: Circ}[A(P);\text{ P}] \\
(\forall x.(Q(x) \vee x = a) \rightarrow (Q(x) \vee x = a) & \\
\quad \wedge \forall x.(Q(x) \vee x = a) \rightarrow P(x)) & \\
\rightarrow (\forall x.(Q(x) \vee x = a) \equiv P(x)) & \text{3. Substitute } Q(x) \vee x = a \text{ for } \phi(x) \\
 & \text{(Universal instantiation in (2).)} \\
\forall x.Q(x) \vee x = a \equiv P(x) & \text{4. Modus Ponens (1, 3)}
\end{array} \tag{18}$$

We have to prove both $\forall x.(Q(x) \vee x = a) \rightarrow P(x)$ and $\forall \phi.[(\forall x.(Q(x) \vee x = a) \rightarrow \phi(x) \wedge \forall x.(\phi(x) \rightarrow P(x))) \rightarrow (\forall x.\phi(x) \equiv P(x))]$ from $\forall x.(Q(x) \vee x = a) \equiv P(x)$. The first is just half of the equivalence, so let's try the second. Let's assume the antecedent, for arbitrary $\phi$:

$\leftarrow$:

$$
\begin{array}{ll}
\forall x.(Q(x) \vee x = a) \equiv P(x) & \text{1. Given} \\
\forall x.(Q(x) \vee x = a) \rightarrow \phi(x) \wedge \forall x.\phi(x) \rightarrow P(x) & \text{2. Assumed} \\
\forall x.(Q(x) \vee x = a) \rightarrow \phi(x) \wedge \forall x.\phi(x) \rightarrow (Q(x) \vee x = a) & \text{3. Substitute for def. of } P(x) \\
\forall x.\phi(x) \equiv (Q(x) \vee x = a) & \text{4. Definition of } \equiv \\
\forall x.\phi(x) \equiv P(x) & \text{5. Transitivity of } \equiv \text{ (1, 4)}
\end{array}
\tag{19}
$$

7. [**5 points**] $A(P) \equiv [P(a) \vee P(b)]$ (This one is tricky! The form won't be exactly like the others.)

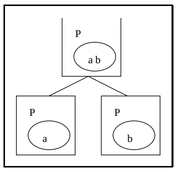
**Solution:**

**Point Breakdown**:

(a) (-2) : The wrong answer, including $\forall x.P(x) \equiv [(x = a \wedge \neg P(b)) \vee (x = a \wedge \neg P(a))]$, $\forall x.[P(x) \equiv x = a \vee P(x) \equiv x = b \wedge \neg \exists y.(P(x) \wedge P(y) \wedge x \neq y)]$, or $\forall x.P(x) \equiv x = a \vee x = b \wedge (\neg P(a) \vee \neg P(b))$,

You can see from the minimal models that there are two possibilities for $P$ – only $a$ is true of $P$, or only $b$ is true of $P$.

$$
Circ[A(P); P] \equiv [\forall x.(x = a) \equiv P(x) \vee \forall x.(x = b) \equiv P(x)]
\tag{20}
$$




Proof:

$\rightarrow$:

$$
\begin{array}{ll}
P(a) \vee P(b) & \text{1. Given from Circ}[A(P)\text{; P}] \\
\forall \phi.[(\phi(a) \vee \phi(b)) \wedge \forall x.\phi(x) \rightarrow P(x)) \rightarrow & \\
\quad (\forall x.\phi(x) \equiv P(x))] & \text{2. Given from: Circ}[A(P)\text{; P}] \\
\forall \phi.[(\phi(a) \wedge \forall x.\phi(x) \rightarrow P(x)) \rightarrow (\forall x.\phi(x) \equiv P(x)) \wedge & \\
(\phi(b) \wedge \forall x.\phi(x) \rightarrow P(x)) \rightarrow (\forall x.\phi(x) \equiv P(x))] & \text{3. } a \vee b \rightarrow c \equiv a \rightarrow c \wedge b \rightarrow c \\
\forall \phi.[(\phi(a) \wedge \forall x.\phi(x) \rightarrow P(x)) \rightarrow (\forall x.\phi(x) \equiv P(x))] & \text{4. } \forall\wedge \equiv \wedge\forall \text{ on (3)} \\
 & \text{(Along with some } \wedge\text{-elimination)} \\
\forall \phi.[(\phi(b) \wedge \forall x.\phi(x) \rightarrow P(x)) \rightarrow (\forall x.\phi(x) \equiv P(x))] & \text{5. } \forall\wedge \equiv \wedge\forall \text{ on (3)} \\
 & \text{(Along with some } \wedge\text{-elimination)} \\
(a = a \wedge \forall x.x = a \rightarrow P(x)) \rightarrow (\forall x.x = a \equiv P(x)) & \text{6. Substitute } x = a \text{ for } \phi(x) \text{ in (4)} \\
(b = b \wedge \forall x.x = b \rightarrow P(x)) \rightarrow (\forall x.x = b \equiv P(x)) & \text{7. Substitute } x = b \text{ for } \phi(x) \text{ in (5)} \\
P(a) \rightarrow (\forall x.x = a \equiv P(x)) & \text{8. Reduction on (6)} \\
P(b) \rightarrow (\forall x.x = b \equiv P(x)) & \text{9. Reduction on (7)} \\
\forall x.x = a \equiv P(x)) \vee \forall x.x = b \equiv P(x)) & \text{10. Modus Ponens. (1, 8, 9)}
\end{array}
\tag{21}
$$

We have to prove both $P(a) \vee P(b)$ and $\forall\phi.[(\phi(a) \vee \phi(b)) \wedge \forall x.(\phi(x) \rightarrow P(x))) \rightarrow (\forall x.\phi(x) \equiv P(x))]$ from $\forall x.(x = a) \equiv P(x) \vee \forall x.(x = b) \equiv P(x)$. The first you can get just from substituting $a$ and $b$ in the formula above. Let's try the second. Let's assume the antecedent, for arbitrary $\phi$:

$$\begin{array}{ll}
\leftarrow: & \\
\quad \forall x.(x = a) \equiv P(x) \vee \forall x.(x = b) \equiv P(x) & \text{1. Given} \\
\quad \forall x.((\phi(a) \vee \phi(b)) \wedge \forall x.\phi(x) \rightarrow P(x) & \text{2. Assumed} \\
\quad \phi(a) \vee \phi(b) & \text{3. } \wedge\text{-elimination on (2)} \\
\quad \forall x.\phi(x) \rightarrow P(x)) & \text{4. } \wedge\text{-elimination (2)}
\end{array} \tag{22}$$

We have two disjunctions, (1) and (3). The easiest thing to do is prove $\phi = P$ by cases:

**Case 1: $\forall x.(x = a) \equiv P(x)$ and $\phi(a)$.**

$$\begin{array}{ll}
\leftarrow: & \\
\quad \forall x.(x = a) \equiv P(x) & \text{5. Assumed (by case 1).} \\
\quad \phi(a) & \text{6. Assumed (by case 1).} \\
\quad \forall x.\phi(x) \rightarrow (x = a) & \text{7. Substitute for def. of P. into (4)} \\
\quad \forall x.x = a \rightarrow \phi(x) & \text{8. Rewrite of (6).} \\
\quad \forall x.x = a \equiv \phi(x) & \text{9. Def. of } \equiv \text{ (7, 8).} \\
\quad \forall x.P(x) \equiv \phi(x) & \text{10. Trans. of } \equiv \text{ (5, 9).}
\end{array} \tag{23}$$

**Case 2: $\forall x.(x = a) \equiv P(x)$ and $\phi(b)$.**

$$\begin{array}{ll}
\leftarrow: & \\
\quad \forall x.(x = a) \equiv P(x) & \text{5. Assumed (by case 2).} \\
\quad \phi(b) & \text{6. Assumed (by case 2).} \\
\quad \forall x.\phi(x) \rightarrow (x = a) & \text{7. Substitute for def. of P into (4).} \\
\quad P(b) & \text{8. Univ. inst. in 4., Modus Ponens with (6).} \\
\quad b = a & \text{9. Subst. into (5) using (8).} \\
\quad \forall x.x = a \rightarrow \phi(x) & \text{10. Rewrite of (6) with (9).} \\
\quad \forall x.x = a \equiv \phi(x) & \text{11. Def. of } \equiv \text{ (7, 10).} \\
\quad \forall x.P(x) \equiv \phi(x) & \text{12. Trans. of } \equiv \text{ (5, 11).}
\end{array} \tag{24}$$

The proofs for cases 3 and 4 are symmetric.

8. [**10 points**] Pick one of the above and prove that the circumscription is equivalent to your answer, as we did above.

**Solution:**

**Point Breakdown**

(a) (-1): Each small error in proof.

(b) (-1): Skipping steps.

(c) (-1): Forgetting to show that the equivalent FOL theory $\rightarrow A(P)$.

(d) (-3): Very strange proof steps, with errors.

(e) (-3): Not proving that $A(P)$ follows from the equivalent first order theory.

(f) (-6): Numerous errors and omissions in proof.

## 2 Problem Set #1 Solutions, section 2.3

1. [**10 points**] Prove: $Circ[A(P,Z);P;Z] \equiv A(P,Z) \wedge Circ[\exists\zeta.A(P,\zeta);P]$ (We can always reduce circumscription with variation to simple circumscription.)

   **Solution:**

   **Point Breakdown**:

   (a) (-1) : For confusion in the proof.

   (b) (-1) : For not being clear on a step.

   (c) (-3) : Numerous errors and confusion.

$$\begin{aligned} Circ[A(P,Z);P;Z] &\equiv A(P,Z) \wedge \forall\phi.[(\exists\zeta.A(\phi,\zeta) \wedge \phi \leq P) \rightarrow (\phi = P)] \\ &\equiv A(P,Z) \wedge \exists\zeta.A(P,\zeta) \wedge \forall\phi.[(\exists\zeta.A(\phi,\zeta) \wedge \phi \leq P) \rightarrow (\phi = P)] \\ &\equiv A(P,Z) \wedge Circ[\exists\zeta.A(\phi,\zeta);P] \end{aligned} \tag{25}$$

   In the second line we introduce $\exists\zeta.A(P,\zeta)$ as $A(P,Z) \equiv A(P,Z) \wedge \exists\zeta.A(P,\zeta)$. $\rightarrow$ because of $\exists$-generalization, and $\leftarrow$ because of $\wedge$-elimination.

2. [**45 points**] Consider the following theory $A(ab)$ about birds:

$$\begin{array}{l} \forall x.(\neg ab(x) \wedge bird(x)) \rightarrow flies(x) \\ bird(Tweety) \end{array} \tag{26}$$

   (a) [**5 points**] Assume that Tweety is the only object in the universe. Draw all the models of this theory, and order them according to $\leq_{ab}$. Can we conclude anything about Tweety flying from the theory $Circ[A(ab);ab]$?

   **Solution:**

   **Point Breakdown**

   i. (-2): For each missing model (Typically, the model $M$ where $[\![Ab]\!] = [\![Bird]\!] = [\![Flies]\!] = \{Tweety\}$. To clarify, this is a model of the theory, but not an ab-minimal one, so it's not a model of $Circ[A(ab);ab]$.)

   If Tweety is the only object in the universe,[1] then the entire theory can be reduced to a propositional one. Thus we can use our old friend the truth table to compute the models:

| $ab(Tweety)$ | $bird(Tweety)$ | $flies(Tweety)$ | satisfies theory 26? |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| **0** | **1** | **1** | **1** |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| **1** | **1** | **0** | **1** |
| **1** | **1** | **1** | **1** |

   The three possible models, ordered according to $\leq_{ab}$ are drawn below:

[1] We can formalize this using the domain closure axiom $\forall x.x = Tweety$

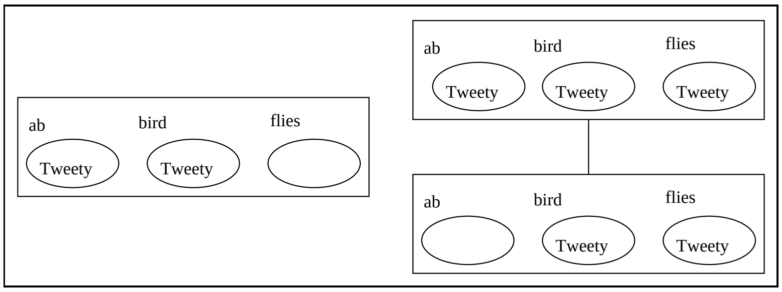



Figure 1: Models of the Bird Theory.

In the two ab-minimal models, we get either that Tweety is a bird who isn't abnormal and can fly, or that he is a bird that is abnormal and can't fly. So since $flies(Tweety)$ doesn't hold in all ab-minimal models, it is not the case that $Circ[A(P); P] \models flies(Tweety)$, and we cannot conclude anything about Tweety flying.

(b) [**5 points**] Draw the models again, this time with ordering $\leq_{ab;flies}$. What can you conclude from the theory $Circ[A(ab); ab; flies]$?

**Solution:**

**Point Breakdown**:

i. (-1) : Wrong reasons for why we conclude $flies(Tweety)$.

The models are still the same, only the ordering is different:

The three possible models, ordered according to $\leq_{ab;flies}$ are drawn below:

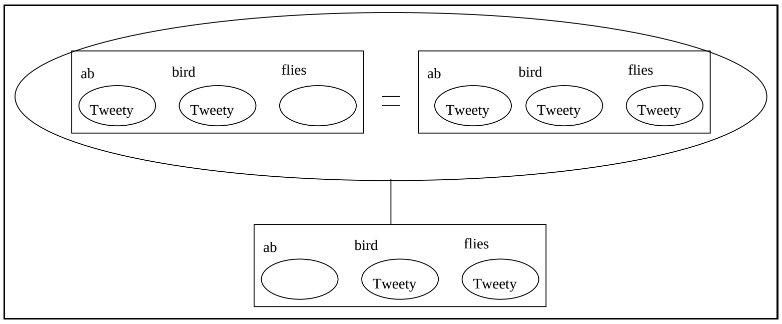



Figure 2: Models of the Bird Theory, with Flies varied.

Now there is only one ab-minimal model, where Tweety is a bird that is not abnormal and flies.

(c) [**20 points**] Find and prove the first order equivalent of $\mathtt{Circ}[\mathtt{A}(\mathtt{ab}, \mathtt{flies}); \mathtt{ab}; \mathtt{flies}]$.

**Solution:**

**Point Breakdown**:

i. (-10): No proof that the first order theory is equivalent.

ii. (-5) : A lot of people thought that the first order equivalent theory was just $\forall x. \neq ab(x)$ or the equivalent. This is not correct! In general, to get the first order equivalent to the circumscription, you need to append the original theory to the definition of $ab$. (This was in the problem set.) How on earth otherwise can you prove statements like $bird(Tweety)$ or $\forall x.(\neg ab(x) \wedge bird(x)) \rightarrow flies(x)$ from just $\forall x.\neg ab(x)$???

iii. (-2,-3) : Erroneous proof, in conjunction with above.

iv. (-1) : Bad proof step.

Well, we said that the circumscription is usually equivalent to the original $A(ab)$, with the definition of $ab$. From the previous question it should be apparent that $\forall x.ab(x) \equiv \perp$. So the first order equivalent theory is (we think):

$$Circ[A(ab, flies); ab; flies] \equiv \begin{array}{l}[\forall x.(\neg ab(x) \wedge bird(x)) \rightarrow flies(x) \\ \wedge bird(Tweety) \\ \wedge \forall x.ab(x) \equiv \perp]\end{array} \tag{27}$$

So let's try proving the equivalence:

$\rightarrow$:

| | |
|---|---|
| $Circ[A(ab, flies); ab; flies]$ | 1. Given. |
| $A(ab, flies) \wedge \forall \phi.[(\exists \zeta.A(\phi, \zeta) \wedge \phi \leq ab) \rightarrow \phi = ab]$ | 2. Def. of Circ |
| $A(ab, flies)$ | 3. $\wedge$-elimination. |
| $\forall \phi.[(\exists \zeta.A(\phi, \zeta) \wedge \phi \leq ab) \rightarrow \phi = ab]$ | 4. $\wedge$-elimination. |
| $\forall x.(\neg ab(x) \wedge bird(x)) \rightarrow flies(x)$ | 5. From $A(ab, flies)$ in (3).* |
| $bird(Tweety)$ | 6. From $A(ab, flies)$ in (3)*. |
| $(\exists \zeta.A(\perp, \zeta) \wedge \perp \leq ab) \rightarrow \perp = ab$ | 7. Substitute $\perp$ for $\phi$ in (4). |
| $(\exists \zeta.A(\perp, \zeta)) \rightarrow \perp = ab$ | 8. $\perp \leq ab$ is always true. |
| $[\exists \zeta.[\forall x.(\neg \perp \wedge bird(x)) \rightarrow \zeta(x)] \wedge bird(Tweety)] \rightarrow \perp = ab$ | 9. Substitute for $A(\perp, \zeta)$. |
| $[\exists \zeta.\forall x.(bird(x)) \rightarrow \zeta(x)] \rightarrow \perp = ab$ | 10. $bird(Tweety)$ is from (6), $\perp \wedge P \equiv P$. |
| $\perp = ab$ | 11. Of course $\exists \zeta.\forall x.(bird(x)) \rightarrow \zeta(x)$ holds - $\zeta$ can be $bird$. |
| $\forall x. \perp \equiv ab(x)$ | 12. Definition of =.* |

(28)

The three conjuncts we needed to prove from $Circ[A(ab, flies); ab; flies]$ are marked with *. Now the other direction: we need to show $A(ab, flies) \wedge \forall \phi.[(\exists \zeta.A(\phi, \zeta) \wedge \phi \leq ab) \rightarrow \phi = ab]$ follows from $A(ab, flies) \wedge \forall x.ab(x) \equiv \perp$. Obviously $A(ab, flies)$ follows from this, but the second part isn't so obvious. To prove this implication, let's assume the antecedent holds, for an arbitrary $\phi$ and $\zeta$.[2] From this we have to show that $\phi = ab$:

[2] Figuring out $\zeta$ must be set to some arbitrary value may not be obvious, until you push the $\exists$ outside the implication to turn it into a $\forall$.

$\leftarrow$:

$$\begin{array}{ll} \forall x.(\neg ab(x) \wedge bird(x)) \rightarrow flies(x) & \text{1. Given.} \\ bird(Tweety) & \text{2. Given.} \\ \forall x.ab(x) \equiv \perp & \text{3. Given.} \\ A(\phi, \zeta) \wedge \phi \leq ab & \text{4. Assumed.} \\ A(\phi, \zeta) & \text{5. } \wedge\text{-elimination.} \\ \phi \leq ab & \text{6. } \wedge\text{-elimination.} \\ \phi \leq \perp & \text{7. Substitute for definition of } ab \text{ in (3).} \\ \phi = \perp & \text{8. From definition of } \leq \text{ and modus tolens.} \\ \phi = ab & \text{9. Transitivity of } = \text{(3, 8).} \end{array} \tag{29}$$

(d) [**5 points**] Go to `http://www.ida.liu.se/labs/kplab/projects/dls/circsimple.html`, and modify the theory to match $A(ab)$. Set "Predicates to be minimized" to $Ab$, and nothing for "Predicates to be varied." (Erase the entire field or it will complain.) Submit the form, and return the output as your answer.

**Solution:**

```
all a


        -Ab(a)
        |
        exists y
              (Bird(y) & -Flies(y) & -Ab(y))
        |
          (Bird(a) & -Flies(a))
        )
```

(e) [**5 points**] The output is only a consequence of the input – it isn't complete. If you combine the output with the original theory above, what are the only two possibilities for Tweety in terms of $bird$, $flies$, and $ab$? (This should be the same as your answer to 2a.)

**Solution:**

**Point Breakdown**:

i. (-1) : For not saying what we could conclude about Tweety.

So let's combine what we got in 2d with our original theory in equation 26. I've moved the "exists y" part outside of the scope of $a$ since it doesn't mention $a$, and renamed $a$ to $x$:

$$\begin{array}{l} \forall x.(\neg ab(x) \wedge bird(x)) \rightarrow flies(x) \wedge \\ bird(Tweety) \wedge \\ (\exists y.(bird(y) \wedge \neg flies(y) \wedge \neg ab(y)) \vee \forall x.\neg ab(x) \vee (bird(x) \wedge \neg flies(x))) \end{array} \tag{30}$$

Clearly, there can't be a bird which is normal but doesn't fly. $\exists y.bird(y \wedge \neg flies(y) \wedge \neg ab(y)$ is false, so we can eliminate it from the disjunction. ($\perp \vee P \equiv P$).

$$\begin{aligned}&\forall x.(\neg ab(x) \land bird(x)) \rightarrow flies(x) \land \\ &bird(Tweety) \land \\ &(\forall x.\neg ab(x) \lor (bird(x) \land \neg flies(x)))\end{aligned} \tag{31}$$

If we plug in for Tweety in the above theory, we find that he is either normal, or doesn't fly. This information combined with first conjunct above gives us two possibilities. Either Tweety flies and is normal, or doesn't fly and is abnormal.

In logic: either $bird(Tweety) \land \neg ab(Tweety) \land flies(Tweety)$, or $bird(Tweety) \land ab(Tweety) \land \neg flies(Tweety)$.

(f) [**5 points**] Now go back and set "Predicates to be varied" to Flies. What's the answer now? What can we conclude about Tweety?

**Solution:**

```
all y
    -Ab(y)
```

If we combine this with our original answer, we get:

$$\begin{aligned}&\forall x.(\neg ab(x) \land bird(x)) \rightarrow flies(x) \land \\ &bird(Tweety) \land \\ &\forall x.\neg ab(x)\end{aligned} \tag{32}$$

which gives us only one possibility, that $bird(Tweety) \land \neg ab(Tweety) \land flies(Tweety)$.