Recursive Functions of Symbolic Expressions and Their Computation by Machine, Part I

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1 Introduction

A programming system called LISP (for LISt Processor) has been developed for the IBM 704 computer by the Artificial Intelligence group at M.I.T. The system was designed to facilitate experiments with a proposed system called the Advice Taker, whereby a machine could be instructed to handle declarative as well as imperative sentences and could exhibit "common sense" in carrying out its instructions. The original proposal [1] for the Advice Taker was made in November 1958. The main requirement was a programming system for manipulating expressions representing formalized declarative and imperative sentences so that the Advice Taker system could make deductions.

In the course of its development the LISP system went through several stages of simplification and eventually came to be based on a scheme for representing the partial recursive functions of a certain class of symbolic expressions. This representation is independent of the IBM 704 computer, or of any other electronic computer, and it now seems expedient to expound the system by starting with the class of expressions called S-expressions and the functions called S-functions.

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In this article, we first describe a formalism for defining functions recursively. We believe this formalism has advantages both as a programming language and as a vehicle for developing a theory of computation. Next, we describe S-expressions and S-functions, give some examples, and then describe the universal S-function *apply* which plays the theoretical role of a universal Turing machine and the practical role of an interpreter. Then we describe the representation of S-expressions in the memory of the IBM 704 by list structures similar to those used by Newell, Shaw and Simon [2], and the representation of S-functions by program. Then we mention the main features of the LISP programming system for the IBM 704. Next comes another way of describing computations with symbolic expressions, and finally we give a recursive function interpretation of flow charts.

We hope to describe some of the symbolic computations for which LISP has been used in another paper, and also to give elsewhere some applications of our recursive function formalism to mathematical logic and to the problem of mechanical theorem proving.

2 Functions and Function Definitions

We shall need a number of mathematical ideas and notations concerning functions in general. Most of the ideas are well known, but the notion of conditional expression is believed to be new[1], and the use of conditional expressions permits functions to be defined recursively in a new and convenient way.

a. *Partial Functions.* A partial function is a function that is defined only on part of its domain. Partial functions necessarily arise when functions are defined by computations because for some values of the arguments the computation defining the value of the function may not terminate. However, some of our elementary functions will be defined as partial functions.

b. *Propositional Expressions and Predicates.* A propositional expression is an expression whose possible values are T (for truth) and F (for falsity). We shall assume that the reader is familiar with the propositional connectives $\wedge$ ("and"), $\vee$ ("or"), and $\neg$ ("not"). Typical propositional expressions are:

[1] reference Kleene

$$x < y$$

$$(x < y) \wedge (b = c)$$

$$\text{x is prime}$$

A predicate is a function whose range consists of the truth values T and F.

c. *Conditional Expressions.* The dependence of truth values on the values of quantities of other kinds is expressed in mathematics by predicates, and the dependence of truth values on other truth values by logical connectives. However, the notations for expressing symbolically the dependence of quantities of other kinds on truth values is inadequate, so that English words and phrases are generally used for expressing these dependences in texts that describe other dependences symbolically. For example, the function |x| is usually defined in words. Conditional expressions are a device for expressing the dependence of quantities on propositional quantities. A conditional expression has the form

$$(p_1 \rightarrow e_1, \cdots, p_n \rightarrow e_n)$$

where the p's are propositional expressions and the e's are expressions of any kind. It may be read, "If p_1 then e_1 otherwise if p_2 then e_2, $\cdots$, otherwise if p_n then e_n," or "p_1 yields $e_1, \cdots, p_n$ yields e_n." [2]

We now give the rules for determining whether the value of

$$(p_1 \rightarrow e_1, \cdots, p_n \rightarrow e_n)$$

is defined, and if so what its value is. Examine the p's from left to right. If a p whose value is T is encountered before any p whose value is undefined is encountered then the value of the conditional expression is the value of the corresponding e (if this is defined). If any undefined p is encountered before

[2] I sent a proposal for conditional expressions to a *CACM* forum on what should be included in Algol 60. Because the item was short, the editor demoted it to a letter to the editor, for which *CACM* subsequently apologized. The notation given here was rejected for Algol 60, because it had been decided that no new mathematical notation should be allowed in Algol 60, and everything new had to be English. The **if** ... **then** ... **else** that Algol 60 adopted was suggested by John Backus.

a true p, or if all p's are false, or if the e corresponding to the first true p is undefined, then the value of the conditional expression is undefined. We now give examples.

$$(1 < 2 \to 4, 1 > 2 \to 3) = 4$$

$$(2 < 1 \to 4, 2 > 1 \to 3, 2 > 1 \to 2) = 3$$

$$(2 < 1 \to 4, T \to 3) = 3$$

$$(2 < 1 \to \frac{0}{0}, T \to 3) = 3$$

$$(2 < 1 \to 3, T \to \frac{0}{0}) \text{ is undefined}$$

$$(2 < 1 \to 3, 4 < 1 \to 4) \text{ is undefined}$$

Some of the simplest applications of conditional expressions are in giving such definitions as

$$|x| = (x < 0 \to -x, T \to x)$$

$$\delta_{ij} = (i = j \to 1, T \to 0)$$

$$sgn(x) = (x < 0 \to -1, x = 0 \to 0, T \to 1)$$

d. *Recursive Function Definitions.* By using conditional expressions we can, without circularity, define functions by formulas in which the defined function occurs. For example, we write

$$n! = (n = 0 \to 1, T \to n \cdot (n-1)!)$$

When we use this formula to evaluate 0! we get the answer 1; because of the way in which the value of a conditional expression was defined, the meaningless expression $0 \cdot$ (0 - 1)! does not arise. The evaluation of 2! according to this definition proceeds as follows:

$$
\begin{aligned}
2! &= (2 = 0 \rightarrow 1, T \rightarrow 2 \cdot (2-1)!) \\
&= 2 \cdot 1! \\
&= 2 \cdot (1 = 0 \rightarrow 1T \rightarrow \cdot (1-1)!) \\
&= 2 \cdot 1 \cdot 0! \\
&= 2 \cdot 1 \cdot (0 = 0 \rightarrow 1, T \rightarrow 0 \cdot (0-1)!) \\
&= 2 \cdot 1 \cdot 1 \\
&= 2
\end{aligned}
$$

We now give two other applications of recursive function definitions. The greatest common divisor, gcd(m,n), of two positive integers m and n is computed by means of the Euclidean algorithm. This algorithm is expressed by the recursive function definition:

$$gcd(m,n) = (m > n \rightarrow gcd(n,m), rem(n,m) = 0 \rightarrow m, T \rightarrow gcd(rem(n,m),m))$$

where $rem(n,m)$ denotes the remainder left when n is divided by m.

The Newtonian algorithm for obtaining an approximate square root of a number a, starting with an initial approximation x and requiring that an acceptable approximation y satisfy $|y^2 - a| < \epsilon$, may be written as

sqrt(a, x, ϵ)

$$= (|x^2 - a| < \epsilon \rightarrow \text{x,T} \rightarrow \text{sqrt (a, } \tfrac{1}{2}(x + \tfrac{a}{x}), \epsilon))$$

The simultaneous recursive definition of several functions is also possible, and we shall use such definitions if they are required.

There is no guarantee that the computation determined by a recursive definition will ever terminate and, for example, an attempt to compute n! from our definition will only succeed if n is a non-negative integer. If the computation does not terminate, the function must be regarded as undefined for the given arguments.

The propositional connectives themselves can be defined by conditional expressions. We write

$$
\begin{aligned}
p \wedge q &= (p \to q, T \to F) \\
p \vee q &= (p \to T, T \to q) \\
\neg p &= (p \to F, T \to T) \\
p \supset q &= (p \to q, T \to T)
\end{aligned}
$$

It is readily seen that the right-hand sides of the equations have the correct truth tables. If we consider situations in which p or q may be undefined, the connectives $\wedge$ and $\vee$ are seen to be noncommutative. For example if p is false and q is undefined, we see that according to the definitions given above $p \wedge q$ is false, but $q \wedge p$ is undefined. For our applications this noncommutativity is desirable, since $p \wedge q$ is computed by first computing p, and if p is false q is not computed. If the computation for p does not terminate, we never get around to computing q. We shall use propositional connectives in this sense hereafter.

e. *Functions and Forms.* It is usual in mathematics—outside of mathematical logic—to use the word "function" imprecisely and to apply it to forms such as $y^2 + x$. Because we shall later compute with expressions for functions, we need a distinction between functions and forms and a notation for expressing this distinction. This distinction and a notation for describing it, from which we deviate trivially, is given by Church [3].

Let f be an expression that stands for a function of two integer variables. It should make sense to write $f(3,4)$ and the value of this expression should be determined. The expression y^2+x does not meet this requirement; $y^2+x(3,4)$ is not a conventional notation, and if we attempted to define it we would be uncertain whether its value would turn out to be 13 or 19. Church calls an expression like $y^2 + x$, a form. A form can be converted into a function if we can determine the correspondence between the variables occurring in the form and the ordered list of arguments of the desired function. This is accomplished by Church's λ-notation.

If $\mathcal{E}$ is a form in variables $x_1, \cdots, x_n$, then $\lambda((x_1, \cdots, x_n), \mathcal{E})$ will be taken to be the function of n variables whose value is determined by substituting the arguments for the variables $x_1, \cdots, x_n$ in that order in $\mathcal{E}$ and evaluating the resulting expression. For example, $\lambda((x,y), y^2 + x)$ is a function of two variables, and $\lambda((x,y), y^2 + x)(3,4) = 19$.

The variables occurring in the list of variables of a λ-expression are dummy or bound, like variables of integration in a definite integral. That is, we may change the names of the bound variables in a function expression without changing the value of the expression, provided that we make the same change for each occurrence of the variable and do not make two variables the same that previously were different. Thus $\lambda((x, y), y^2 + x), \lambda((u, v), v^2 + u)$ and $\lambda((y, x), x^2 + y)$ denote the same function.

We shall frequently use expressions in which some of the variables are bound by λ's and others are not. Such an expression may be regarded as defining a function with parameters. The unbound variables are called free variables.

An adequate notation that distinguishes functions from forms allows an unambiguous treatment of functions of functions. It would involve too much of a digression to give examples here, but we shall use functions with functions as arguments later in this report.

Difficulties arise in combining functions described by λ-expressions, or by any other notation involving variables, because different bound variables may be represented by the same symbol. This is called collision of bound variables. There is a notation involving operators that are called combinators for combining functions without the use of variables. Unfortunately, the combinatory expressions for interesting combinations of functions tend to be lengthy and unreadable.

f. *Expressions for Recursive Functions.* The λ-notation is inadequate for naming functions defined recursively. For example, using λ's, we can convert the definition

$$\mathrm{sqrt}(a, x, \epsilon) = (|x^2 - a| < \epsilon \rightarrow x, T \rightarrow \mathrm{sqrt}(a, \frac{1}{2}(x + \frac{a}{x}), \epsilon))$$

into

$$\mathrm{sqrt} = \lambda((a, x, \epsilon), (|x^2 - a| < \epsilon \rightarrow x, T \rightarrow \mathrm{sqrt}(a, \frac{1}{2}(x + \frac{a}{x}), \epsilon))),$$

but the right-hand side cannot serve as an expression for the function because there would be nothing to indicate that the reference to *sqrt* within the expression stood for the expression as a whole.

In order to be able to write expressions for recursive functions, we introduce another notation. $label(a, \mathcal{E})$ denotes the expression $\mathcal{E}$, provided that occurrences of a within $\mathcal{E}$ are to be interpreted as referring to the expression as a whole. Thus we can write

$$\text{label}(\text{sqrt}, \ \lambda((a,x,\epsilon),(|x^2-a|<\epsilon \rightarrow x, T \rightarrow \text{sqrt}(a, \tfrac{1}{2}(x+\tfrac{a}{x}),\epsilon))))$$

as a name for our sqrt function.

The symbol a in label $(a, \mathcal{E})$ is also bound, that is, it may be altered systematically without changing the meaning of the expression. It behaves differently from a variable bound by a λ, however.

3 Recursive Functions of Symbolic Expressions

We shall first define a class of symbolic expressions in terms of ordered pairs and lists. Then we shall define five elementary functions and predicates, and build from them by composition, conditional expressions, and recursive definitions an extensive class of functions of which we shall give a number of examples. We shall then show how these functions themselves can be expressed as symbolic expressions, and we shall define a universal function *apply* that allows us to compute from the expression for a given function its value for given arguments. Finally, we shall define some functions with functions as arguments and give some useful examples.

a. *A Class of Symbolic Expressions.* We shall now define the S-expressions (S stands for symbolic). They are formed by using the special characters

$$\cdot$$
$$)$$
$$($$

and an infinite set of distinguishable atomic symbols. For atomic symbols, we shall use strings of capital Latin letters and digits with single imbedded blanks.[3] Examples of atomic symbols are

$$\begin{array}{l} A \\ ABA \\ APPLE\ PIE\ NUMBER\ 3 \end{array}$$

There is a twofold reason for departing from the usual mathematical practice of using single letters for atomic symbols. First, computer programs frequently require hundreds of distinguishable symbols that must be formed from the 47 characters that are printable by the IBM 704 computer. Second, it is convenient to allow English words and phrases to stand for atomic entities for mnemonic reasons. The symbols are atomic in the sense that any substructure they may have as sequences of characters is ignored. We assume only that different symbols can be distinguished. S-expressions are then defined as follows:

1. Atomic symbols are S-expressions.
2. If e_1 and e_2 are S-expressions, so is $(e_1 \cdot e_2)$.

Examples of S-expressions are

$$\begin{array}{l} AB \\ (A \cdot B) \\ ((AB \cdot C) \cdot D) \end{array}$$

An S-expression is then simply an ordered pair, the terms of which may be atomic symbols or simpler S-expressions. We can can represent a list of arbitrary length in terms of S-expressions as follows. The list

$$(m_1, m_2, \cdots, m_n)$$

is represented by the S-expression

$$(m_1 \cdot (m_2 \cdot (\cdots (m_n \cdot NIL) \cdots)))$$

Here NIL is an atomic symbol used to terminate lists. Since many of the symbolic expressions with which we deal are conveniently expressed as lists, we shall introduce a list notation to abbreviate certain S-expressions. We have

[3] 1995 remark: Imbedded blanks could be allowed within symbols, because lists were then written with commas between elements.

l. (m) stands for (m $\cdot NIL$).
2. $(m_1, \cdots, m_n)$ stands for $(m_1 \cdot (\cdots (m_n \cdot NIL) \cdots))$.
3. $(m_1, \cdots, m_n \cdot x)$ stands for $(m_1 \cdot (\cdots (m_n \cdot x) \cdots))$.

Subexpressions can be similarly abbreviated. Some examples of these abbreviations are
$((AB, C), D)$ for $((AB \cdot (C \cdot NIL)) \cdot (D \cdot NIL))$
$((A, B), C, D \cdot E)$ for $((A \cdot (B \cdot NIL)) \cdot (C \cdot (D \cdot E)))$

Since we regard the expressions with commas as abbreviations for those not involving commas, we shall refer to them all as S-expressions.

b. *Functions of S-expressions and the Expressions That Represent Them.* We now define a class of functions of S-expressions. The expressions representing these functions are written in a conventional functional notation. However, in order to clearly distinguish the expressions representing functions from S-expressions, we shall use sequences of lower-case letters for function names and variables ranging over the set of S-expressions. We also use brackets and semicolons, instead of parentheses and commas, for denoting the application of functions to their arguments. Thus we write

$$car[x]$$
$$car[cons[(A \cdot B); x]]$$

In these M-expressions (meta-expressions) any S-expression that occur stand for themselves.

c. *The Elementary S-functions and Predicates.* We introduce the following functions and predicates:
1. atom. atom[x] has the value of T or F according to whether x is an atomic symbol. Thus

atom [X] = T
atom [(X $\cdot$ A)] = F

2. eq. eq [x;y] is defined if and only if both x and y are atomic. eq [x; y] = T if x and y are the same symbol, and eq [x; y] = F otherwise. Thus

eq [X; X] = T
eq [X; A] = F
eq [X; (X · A)] is undefined.

3. car. car[x] is defined if and only if x is not atomic. car $[(e_1 \cdot e_2)] = e_1$. Thus car [X] is undefined.

car $[(X \cdot A)]$ = X
car $[((X \cdot A) \cdot Y)] = (X \cdot A)$

4. cdr. cdr [x] is also defined when x is not atomic. We have cdr $[(e_1 \cdot e_2)] = e_2$. Thus cdr [X] is undefined.

cdr $[(X \cdot A)]$ = A cdr $[((X \cdot A) \cdot Y)]$ = Y

5. cons. cons [x; y] is defined for any x and y. We have cons $[e_1; e_2] = (e_1 \cdot e_2)$. Thus

cons [X; A] = (X A)
cons $[(X \cdot A); Y] = ((X \cdot A)Y)$

car, cdr, and cons are easily seen to satisfy the relations

car [cons [x; y]] = x
cdr [cons [x; y]] = y
cons [car [x]; cdr [x]] = x, provided that x is not atomic.

The names "car" and "cons" will come to have mnemonic significance only when we discuss the representation of the system in the computer. Compositions of car and cdr give the subexpressions of a given expression in a given position. Compositions of cons form expressions of a given structure out of parts. The class of functions which can be formed in this way is quite limited and not very interesting.

d. *Recursive S-functions.* We get a much larger class of functions (in fact, all computable functions) when we allow ourselves to form new functions of S-expressions by conditional expressions and recursive definition. We now give some examples of functions that are definable in this way.

1. ff$[x]$. The value of ff$[x]$ is the first atomic symbol of the S-expression x with the parentheses ignored. Thus

$$\mathrm{ff}[((A\cdot B)\cdot C)] = A$$

We have

$$\mathrm{ff}[x] = [atom[x] \to x; T \to \mathrm{ff}[car[x]]]$$

We now trace in detail the steps in the evaluation of

ff [((A · B) · C)]:
ff [((A · B) · C)]

$$
\begin{aligned}
&= [atom[((A\cdot B)\cdot C)] \to ((A\cdot B)\cdot C); T \to \mathrm{ff}[car[((A\cdot B)C\cdot)]]] \\
&= [F \to ((A\cdot B)\cdot C); T \to \mathrm{ff}[car[((A\cdot B)\cdot C)]]] \\
&= [T \to \mathrm{ff}[car[((A\cdot B)\cdot C)]]] \\
&= \mathrm{ff}[car[((A\cdot B)\cdot C)]] \\
&= \mathrm{ff}[(A\cdot B)] \\
&= [atom[(A\cdot B)] \to (A\cdot B); T \to \mathrm{ff}[car[(A\cdot B)]]] \\
&= [F \to (A\cdot B); T \to \mathrm{ff}[car[(A\cdot B)]]] \\
&= [T \to \mathrm{ff}[car[(A\cdot B)]]] \\
&= \mathrm{ff}[car[(A\cdot B)]] \\
&= \mathrm{ff}[A]
\end{aligned}
$$

$$= \quad [atom[A] \rightarrow A; T \rightarrow \text{ff}[car[A]]]$$

$$= \quad [T \rightarrow A; T \rightarrow \text{ff}[car[A]]]$$

$$= \quad A$$

2. subst $[x; y; z]$. This function gives the result of substituting the S-expression x for all occurrences of the atomic symbol y in the S-expression z. It is defined by

subst [x; y; z] = [atom [z] → [eq [z; y] → x; T → z];
T → cons [subst [x; y; car [z]]; subst [x; y; cdr [z]]]]

As an example, we have

$$subst[(X \cdot A); B; ((A \cdot B) \cdot C)] = ((A \cdot (X \cdot A)) \cdot C)$$

3. equal [x; y]. This is a predicate that has the value T if x and y are the same S-expression, and has the value F otherwise. We have

equal [x; y] = [atom [x] ∧ atom [y] ∧ eq [x; y]]

∨[¬ atom [x] ∧¬ atom [y] ∧ equal [car [x]; car [y]]

∧ equal [cdr [x]; cdr [y]]]

It is convenient to see how the elementary functions look in the abbreviated list notation. The reader will easily verify that

(i) $car[(m_1, m_2, \cdots, m_n)] = m_1$
(ii) $cdr[(m_s, m_2, \cdots, m_n)] = (m_2, \cdots, m_n)$
(iii) $cdr[(m)] = NIL$
(iv) $cons[m_1; (m_2, \cdots, m_n)] = (m_1, m_2, \cdots, m_n)$
(v) $cons[m; NIL] = (m)$

We define

$null[x] = atom[x] \wedge eq[x; NIL]$
This predicate is useful in dealing with lists.

Compositions of car and cdr arise so frequently that many expressions can be written more concisely if we abbreviate

$cadr[x]$ for $car[cdr[x]]$,
$caddr[x]$ for $car[cdr[cdr[x]]]$, etc.

Another useful abbreviation is to write list $[e_1; e_2; \cdots; e_n]$
for $cons[e_1; cons[e_2; \cdots; cons[e_n; NIL] \cdots]]$.

This function gives the list, $(e_1, \cdots, e_n)$, as a function of its elements.

The following functions are useful when S-expressions are regarded as lists.

1. append [x;y].

append [x; y] = [null[x] → y; T → cons [car [x]; append [cdr [x]; y]]]

An example is

append [(A, B); (C, D, E)] = (A, B, C, D, E)

2. among [x;y]. This predicate is true if the S-expression x occurs among the elements of the list y. We have

$$among[x; y] = \neg null[y] \wedge [equal[x; car[y]] \vee among[x; cdr[y]]]$$

3. pair [x;y]. This function gives the list of pairs of corresponding elements of the lists x and y. We have

$$pair[x; y] = [null[x] \wedge null[y] \to NIL; \neg atom[x] \wedge \neg atom[y] \to cons[list[car[x]; car[y]]; pair[cdr[x]; cdr[y]]]$$

An example is

$$pair[(A, B, C); (X, (Y, Z), U)] = ((A, X), (B, (Y, Z)), (C, U)).$$

4. assoc [x;y]. If y is a list of the form $((u_1, v_1), \cdots, (u_n, v_n))$ and x is one of the u's, then assoc $[x; y]$ is the corresponding v. We have

$$assoc[x; y] = eq[caar[y]; x] \rightarrow cadar[y]; T \rightarrow assoc[x; cdr[y]]]$$

An example is

$$assoc[X; ((W, (A, B)), (X, (C, D)), (Y, (E, F)))] = (C, D).$$

5. $sublis[x; y]$. Here x is assumed to have the form of a list of pairs $((u_1, v_1), \cdots, (u_n, v_n))$, where the u's are atomic, and y may be any S-expression. The value of $sublis[x; y]$ is the result of substituting each v for the corresponding u in y. In order to define sublis, we first define an auxiliary function. We have

$$sub2[x; z] = [null[x] \rightarrow z; eq[caar[x]; z] \rightarrow cadar[x]; T \rightarrow sub2[cdr[x]; z]]$$

and

$$sublis[x; y] = [atom[y] \rightarrow sub2[x; y]; T \rightarrow cons[sublis[x; car[y]]; sublis[x; cdr[y]]]$$

We have

sublis [((X, (A, B)), (Y, (B, C))); (A, X · Y)] = (A, (A, B), B, C)

e. *Representation of S-Functions by S-Expressions.* S-functions have been described by M-expressions. We now give a rule for translating M-expressions into S-expressions, in order to be able to use S-functions for making certain computations with S-functions and for answering certain questions about S-functions.

The translation is determined by the following rules in rich we denote the translation of an M-expression $\mathcal{E}$ by $\mathcal{E}^*$.

1. If $\mathcal{E}$ is an S-expression $\mathcal{E}^*$ is (QUOTE, $\mathcal{E}$).

2. Variables and function names that were represented by strings of lowercase letters are translated to the corresponding strings of the corresponding uppercase letters. Thus car* is CAR, and subst* is SUBST.

3. A form $f[e_1; \cdots; e_n]$ is translated to $(f^*, e_1^* \cdots, e_n^*)$. Thus cons [car [x]; cdr [x]]* is (CONS, (CAR, X), (CDR, X)).

4. $\{[p_1 \rightarrow e_1; \cdots; p_n \rightarrow e_n]\}^*$ is (COND, $(p_1^*, e_1^*), \cdots, (p_n^* \cdot e_n^*)$).

5. $\{\lambda[[x_1; \cdots; x_n]; \mathcal{E}]\}^*$ is (LAMBDA, $(x_1^*, \cdots, x_n^*), \mathcal{E}^*$).

6. $\{label[a; \mathcal{E}]\}^*$ is (LABEL, a*, $\mathcal{E}^*$).

With these conventions the substitution function whose M-expression is label [subst; λ [[x; y; z]; [atom [z] → [eq [y; z] → x; T → z]; T → cons [subst [x; y; car [z]]; subst [x; y; cdr [z]]]]]] has the S-expression

(LABEL, SUBST, (LAMBDA, (X, Y, Z), (COND ((ATOM, Z), (COND, (EQ, Y, Z), X), ((QUOTE, T), Z))), ((QUOTE, T), (CONS, (SUBST, X, Y, (CAR Z)), (SUBST, X, Y, (CDR, Z)))))))

This notation is writable and somewhat readable. It can be made easier to read and write at the cost of making its structure less regular. If more characters were available on the computer, it could be improved considerably.[4]

f. *The Universal S-Function apply.* There is an S-function *apply* with the property that if f is an S-expression for an S-function f' and $args$ is a list of arguments of the form $(arg_1, \cdots, arg_n)$, where $arg_1, \cdots, arg_n$ are arbitrary S-expressions, then $apply[f; args]$ and $f'[arg_1; \cdots; arg_n]$ are defined for the same values of $arg_1, \cdots, arg_n$, and are equal when defined. For example,

$$\lambda[[x; y]; cons[car[x]; y]][(A, B); (C, D)]$$

$$= apply[(LAMBDA, (X, Y), (CONS, (CAR, X), Y)); ((A, B), (C, D))] = (A, C, D)$$

The S-function *apply* is defined by

$$apply[f; args] = eval[cons[f; appq[args]]; NIL],$$

where

$$appq[m] = [null[m] \rightarrow NIL; T \rightarrow cons[list[QUOTE; car[m]]; appq[cdr[m]]]]$$

and

eval[e; a] = [

[4] 1995: More characters were made available on SAIL and later on the Lisp machines. Alas, the world went back to inferior character sets again—though not as far back as when this paper was written in early 1959.

atom [e] → assoc [e; a];

atom [car [e]] → [

eq [car [e]; QUOTE] → cadr [e];

eq [car [e]; ATOM] → atom [eval [cadr [e]; a]];

eq [car [e]; EQ] → [eval [cadr [e]; a] = eval [caddr [e]; a]];

eq [car [e]; COND] → evcon [cdr [e]; a];

eq [car [e]; CAR] → car [eval [cadr [e]; a]];

eq [car [e]; CDR] → cdr [eval [cadr [e]; a]];

eq [car [e]; CONS] → cons [eval [cadr [e]; a]; eval [caddr [e];

a]]; T → eval [cons [assoc [car [e]; a];

evlis [cdr [e]; a]]; a]];

eq [caar [e]; LABEL] → eval [cons [caddar [e]; cdr [e]];

cons [list [cadar [e]; car [e]; a]];

eq [caar [e]; LAMBDA] → eval [caddar [e];

append [pair [cadar [e]; evlis [cdr [e]; a]; a]]]

and

$$evcon[c; a] = [eval[caar[c]; a] \rightarrow eval[cadar[c]; a]; T \rightarrow evcon[cdr[c]; a]]$$

and

$$evlis[m; a] = [null[m] \rightarrow NIL; T \rightarrow cons[eval[car[m]; a]; evlis[cdr[m]; a]]]$$

We now explain a number of points about these definitions. [5]

1. *apply* itself forms an expression representing the value of the function applied to the arguments, and puts the work of evaluating this expression onto a function *eval*. It uses *appq* to put quotes around each of the arguments, so that *eval* will regard them as standing for themselves.

2. $eval[e; a]$ has two arguments, an expression e to be evaluated, and a list of pairs a. The first item of each pair is an atomic symbol, and the second is the expression for which the symbol stands.

3. If the expression to be evaluated is atomic, eval evaluates whatever is paired with it first on the list a.

4. If e is not atomic but $car[e]$ is atomic, then the expression has one of the forms $(QUOTE, e)$ or $(ATOM, e)$ or (EQ, e_1, e_2) or $(COND, (p_1, e_1), \cdots, (p_n, e_n))$, or (CAR, e) or (CDR, e) or $(CONS, e_1, e_2)$ or $(f, e_1, \cdots, e_n)$ where f is an atomic symbol.

In the case $(QUOTE, e)$ the expression e, itself, is taken. In the case of $(ATOM, e)$ or (CAR, e) or (CDR, e) the expression e is evaluated and the appropriate function taken. In the case of (EQ, e_1, e_2) or $(CONS, e_1, e_2)$ two expressions have to be evaluated. In the case of $(COND, (p_1, e_1), \cdots (p_n, e_n))$ the p's have to be evaluated in order until a true p is found, and then the corresponding e must be evaluated. This is accomplished by *evcon*. Finally, in the case of $(f, e_1, \cdots, e_n)$ we evaluate the expression that results from replacing f in this expression by whatever it is paired with in the list a.

5. The evaluation of $((LABEL, f, \mathcal{E}), e_1, \cdots, e_n)$ is accomplished by evaluating $(\mathcal{E}, e_1, \cdots, e_n)$ with the pairing $(f, (LABEL, f, \mathcal{E}))$ put on the front of the previous list a of pairs.

6. Finally, the evaluation of $((LAMBDA, (x_1, \cdots, x_n), \mathcal{E}), e_1, \cdots e_n)$ is accomplished by evaluating $\mathcal{E}$ with the list of pairs $((x_1, e_1), \cdots, ((x_n, e_n))$ put on the front of the previous list a.

The list a could be eliminated, and LAMBDA and LABEL expressions evaluated by substituting the arguments for the variables in the expressions $\mathcal{E}$. Unfortunately, difficulties involving collisions of bound variables arise, but they are avoided by using the list a.

[5] 1995: This version isn't quite right. A comparison of this and other versions of *eval* including what was actually implemented (and debugged) is given in "The Influence of the Designer on the Design" by Herbert Stoyan and included in *Artificial Intelligence and Mathematical Theory of Computation: Papers in Honor of John McCarthy*, Vladimir Lifschitz (ed.), Academic Press, 1991

Calculating the values of functions by using *apply* is an activity better suited to electronic computers than to people. As an illustration, however, we now give some of the steps for calculating
apply [(LABEL, FF, (LAMBDA, (X), (COND, (ATOM, X), X), ((QUOTE, T),(FF, (CAR, X))))));((A· B))] = A
The first argument is the S-expression that represents the function ff defined in section 3d. We shall abbreviate it by using the letter ϕ. We have

apply [ϕ; ((A·B))]
= eval [((LABEL, FF, ψ), (QUOTE, (A·B))); NIL]

where ψ is the part of ϕ beginning (LAMBDA

= eval[((LAMBDA, (X), ω), (QUOTE, (A·B)));((FF, ϕ))]

where ω is the part of ψ beginning (COND

= eval [(COND, $(\pi_1, \epsilon_1), (\pi_2, \epsilon_2)$); ((X, (QUOTE, (A·B))), (FF, ϕ))]

Denoting ((X, (QUOTE, (A·B))), (FF, ϕ)) by a, we obtain

= evcon [((π_1, ϵ_1), (π_2, ϵ_2)); a]

This involves eval [$\pi_1; a$]

= eval [(ATOM, X); a]

= atom [eval [X; a]]

= atom [eval [assoc [X; ((X, (QUOTE, (A·B))), (FF,ϕ))];a]]

= atom [eval [(QUOTE, (A·B)); a]]

= atom [(A·B)],

= F

Our main calulation continues with

apply [ϕ; ((A·B))]

= evcon $[((\pi_2, \epsilon_2,)); a]$,

which involves eval $[\pi_2; a]$ = eval [(QUOTE, T); a] = T

Our main calculation again continues with

apply [ϕ; ((A·B))]

= eval $[\epsilon_2; a]$

= eval [(FF, (CAR, X));a]

= eval [Cons [ϕ; evlis [((CAR, X)); a]]; a]

Evaluating evlis [((CAR, X)); a] involves

eval [(CAR, X); a]

= car [eval [X; a]]

= car [(A·B)], where we took steps from the earlier computation of atom [eval [X; a]] = A,

and so evlis [((CAR, X)); a] then becomes

list [list [QUOTE; A]] = ((QUOTE, A)),

and our main quantity becomes

= eval [(ϕ, (QUOTE, A)); a]

The subsequent steps are made as in the beginning of the calculation. The LABEL and LAMBDA cause new pairs to be added to a, which gives a new list of pairs a_1. The π_1 term of the conditional eval [(ATOM, X); a_1] has the value T because X is paired with (QUOTE, A) first in a_1, rather than with (QUOTE, (A·B)) as in a.

Therefore we end up with eval [X; a_1] from the *evcon*, and this is just A.

g. *Functions with Functions as Arguments.* There are a number of useful functions some of whose arguments are functions. They are especially useful in defining other functions. One such function is $maplist[x; f]$ with an S-expression argument x and an argument f that is a function from S-expressions to S-expressions. We define

$$maplist[x; f] = [null[x] \rightarrow NIL; T \rightarrow cons[f[x]; maplist[cdr[x]; f]]]$$

The usefulness of *maplist* is illustrated by formulas for the partial derivative with respect to x of expressions involving sums and products of x and other variables. The S-expressions that we shall differentiate are formed as follows.

1. An atomic symbol is an allowed expression.

2. If $e_1, e_2, \cdots, e_n$ are allowed expressions, (PLUS, $e_1, \cdots, e_n$) and (TIMES, $e_1, \cdots, e_n$) are also, and represent the sum and product, respectively, of $e_1, \cdots, e_n$.

This is, essentially, the Polish notation for functions, except that the inclusion of parentheses and commas allows functions of variable numbers of arguments. An example of an allowed expression is (TIMES, X, (PLUS, X, A), Y), the conventional algebraic notation for which is X(X + A)Y.

Our differentiation formula, which gives the derivative of y with respect to x, is

diff [y; x] = [atom [y] → [eq [y; x] → ONE; T → ZERO]; eq [car [Y]; PLUS] → cons [PLUS; maplist [cdr [y]; λ[[z]; diff [car [z]; x]]]]; eq[car [y]; TIMES] → cons[PLUS; maplist[cdr[y]; λ[[z]; cons [TIMES; maplist[cdr [y]; λ[[w]; ¬ eq [z; w] → car [w]; T → diff [car [[w]; x]]]]]]]

The derivative of the expression (TIMES, X, (PLUS, X, A), Y), as computed by this formula, is

(PLUS, (TIMES, ONE, (PLUS, X, A), Y), (TIMES, X, (PLUS, ONE, ZERO), Y), (TIMES, X, (PLUS, X, A), ZERO))

Besides *maplist*, another useful function with functional arguments is *search*, which is defined as

$$search[x; p; f; u] = [null[x] \rightarrow u; p[x] \rightarrow f[x]; T \rightarrow search[cdr[x]; p; f; u]$$

The function *search* is used to search a list for an element that has the property p, and if such an element is found, f of that element is taken. If there is no such element, the function u of no arguments is computed.

4 The LISP Programming System

The LISP programming system is a system for using the IBM 704 computer to compute with symbolic information in the form of S-expressions. It has been or will be used for the following purposes:

l. Writing a compiler to compile LISP programs into machine language.

2. Writing a program to check proofs in a class of formal logical systems.

3. Writing programs for formal differentiation and integration.

4. Writing programs to realize various algorithms for generating proofs in predicate calculus.

5. Making certain engineering calculations whose results are formulas rather than numbers.

6. Programming the Advice Taker system.

The basis of the system is a way of writing computer programs to evaluate S-functions. This will be described in the following sections.

In addition to the facilities for describing S-functions, there are facilities for using S-functions in programs written as sequences of statements along the lines of FORTRAN (4) or ALGOL (5). These features will not be described in this article.

a. *Representation of S-Expressions by List Structure.* A list structure is a collection of computer words arranged as in figure 1a or 1b. Each word of the list structure is represented by one of the subdivided rectangles in the figure. The $left$ box of a rectangle represents the *address* field of the word and the $right$ box represents the *decrement* field. An arrow from a box to another rectangle means that the field corresponding to the box contains the location of the word corresponding to the other rectangle.

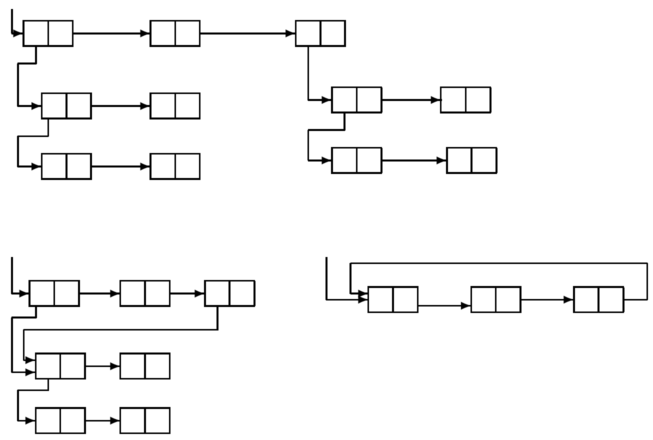
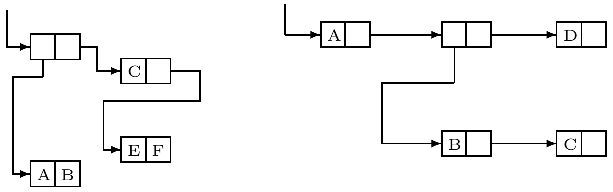


Fig. 1

It is permitted for a substructure to occur in more than one place in a list structure, as in figure 1b, but it is not permitted for a structure to have cycles, as in figure 1c. An atomic symbol is represented in the computer by a list structure of special form called the *association list* of the symbol. The address field of the first word contains a special constant which enables the program to tell that this word represents an atomic symbol. We shall describe association lists in section 4b.

An S-expression x that is not atomic is represented by a word, the address and decrement parts of which contain the locations of the subexpressions $car[x]$ and $cdr[x]$, respectively. If we use the symbols A, B, etc. to denote the locations of the association list of these symbols, then the S-expression $((A \cdot B) \cdot (C \cdot (E \cdot F)))$ is represented by the list structure a of figure 2. Turning to the list form of S-expressions, we see that the S-expression $(A, (B, C), D)$, which is an abbreviation for $(A \cdot ((B \cdot (C \cdot NIL)) \cdot (D \cdot NIL)))$, is represented by the list structure of figure 2b.

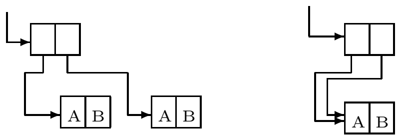


(a) (b)

Figure 2

When a list structure is regarded as representing a list, we see that each term of the list occupies the address part of a word, the decrement part of which *points* to the word containing the next term, while the last word has NIL in its decrement.

An expression that has a given subexpression occurring more than once can be represented in more than one way. Whether the list structure for the subexpression is or is not repeated depends upon the history of the program. Whether or not a subexpression is repeated will make no difference in the results of a program as they appear outside the machine, although it will affect the time and storage requirements. For example, the S-expression ((A·B)·(A·B)) can be represented by either the list structure of figure 3a or 3b.



(a) (b)

Figure 3

The prohibition against circular list structures is essentially a prohibition against an expression being a subexpression of itself. Such an expression could not exist on paper in a world with our topology. Circular list structures would have some advantages in the machine, for example, for representing recursive functions, but difficulties in printing them, and in certain other operations, make it seem advisable not to use them for the present.

The advantages of list structures for the storage of symbolic expressions are:

1. The size and even the number of expressions with which the program will have to deal cannot be predicted in advance. Therefore, it is difficult to arrange blocks of storage of fixed length to contain them.

2. Registers can be put back on the free-storage list when they are no longer needed. Even one register returned to the list is of value, but if expressions are stored linearly, it is difficult to make use of blocks of registers of odd sizes that may become available.

3. An expression that occurs as a subexpression of several expressions need be represented in storage only once.

b. *Association Lists*[6] . In the LISP programming system we put more in the association list of a symbol than is required by the mathematical system described in the previous sections. In fact, any information that we desire to associate with the symbol may be put on the association list. This information may include: the *print name*, that is, the string of letters and digits which represents the symbol outside the machine; a numerical value if the symbol represents a number; another S-expression if the symbol, in some way, serves as a name for it; or the location of a routine if the symbol represents a function for which there is a machine-language subroutine. All this implies that in the machine system there are more primitive entities than have been described in the sections on the mathematical system.

For the present, we shall only describe how *print names* are represented on association lists so that in reading or printing the program can establish a correspondence between information on punched cards, magnetic tape or printed page and the list structure inside the machine. The association list of the symbol DIFFERENTIATE has a segment of the form shown in figure 4. Here *pname* is a symbol that indicates that the structure for the print name of the symbol whose association list this is hanging from the next word on the association list. In the second row of the figure we have a list of three words. The address part of each of these words points to a Word containing

[6] 1995: These were later called property lists.

six 6-bit characters. The last word is filled out with a 6-bit combination that does not represent a character printable by the computer. (Recall that the IBM 7O4 has a 36-bit word and that printable characters are each represented by 6 bits.) The presence of the words with character information means that the association lists do not themselves represent S-expressions, and that only some of the functions for dealing with S-expressions make sense within an association list.

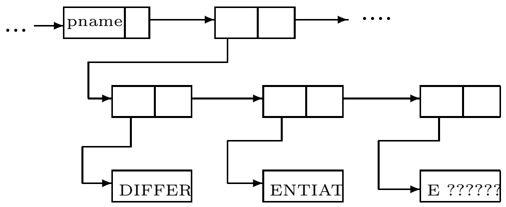


Figure 4

c. *Free-Storage List.* At any given time only a part of the memory reserved for list structures will actually be in use for storing S-expressions. The remaining registers (in our system the number, initially, is approximately 15,000) are arranged in a single list called the *free-storage list.* A certain register, FREE, in the program contains the location of the first register in this list. When a word is required to form some additional list structure, the first word on the *free-storage list* is taken and the number in register FREE is changed to become the location of the second word on the free-storage list. No provision need be made for the user to program the return of registers to the free-storage list.

This return takes place automatically, approximately as follows (it is necessary to give a simplified description of this process in this report): There is a fixed set of base registers in the program which contains the locations of list structures that are accessible to the program. Of course, because list structures branch, an arbitrary number of registers may be involved. Each register that is accessible to the program is accessible because it can be reached from one or more of the base registers by a chain of *car* and *cdr* operations. When the contents of a base register are changed, it may happen that the register to which the base register formerly pointed cannot be reached by a $car - cdr$ chain from any base register. Such a register may be considered abandoned by the program because its contents can no longer be found by any possible program; hence its contents are no longer of interest, and so we would like to have it back on the free-storage list. This comes about in the following way.

Nothing happens until the program runs out of free storage. When a free register is wanted, and there is none left on the free-storage list, a reclamation[7] cycle starts.

First, the program finds all registers accessible from the base registers and makes their signs negative. This is accomplished by starting from each of the base registers and changing the sign of every register that can be reached from it by a $car - cdr$ chain. If the program encounters a register in this process which already has a negative sign, it assumes that this register has already been reached.

After all of the accessible registers have had their signs changed, the program goes through the area of memory reserved for the storage of list structures and puts all the registers whose signs were not changed in the previous step back on the free-storage list, and makes the signs of the accessible registers positive again.

This process, because it is entirely automatic, is more convenient for the programmer than a system in which he has to keep track of and erase unwanted lists. Its efficiency depends upon not coming close to exhausting the available memory with accessible lists. This is because the reclamation process requires several seconds to execute, and therefore must result in the addition of at least several thousand registers to the free-storage list if the program is not to spend most of its time in reclamation.

d. *Elementary S-Functions in the Computer.* We shall now describe the computer representations of *atom*, $=$, *car*, *cdr*, and *cons*. An S-expression is communicated to the program that represents a function as the location of the word representing it, and the programs give S-expression answers in the same form.

atom. As stated above, a word representing an atomic symbol has a special

[7] We already called this process "garbage collection", but I guess I chickened out of using it in the paper—or else the Research Laboratory of Electronics grammar ladies wouldn't let me.

constant in its address part: *atom* is programmed as an open subroutine that tests this part. Unless the M-expression $atom[e]$ occurs as a condition in a conditional expression, the symbol T or F is generated as the result of the test. In case of a conditional expression, a conditional transfer is used and the symbol T or F is not generated.

eq. The program for $eq[e; f]$ involves testing for the numerical equality of the locations of the words. This works because each atomic symbol has only one association list. As with *atom*, the result is either a conditional transfer or one of the symbols T or F.

car. Computing $car[x]$ involves getting the contents of the address part of register x. This is essentially accomplished by the single instruction CLA 0, i, where the argument is in index register, and the result appears in the address part of the accumulator. (We take the view that the places from which a function takes its arguments and into which it puts its results are prescribed in the definition of the function, and it is the responsibility of the programmer or the compiler to insert the required datamoving instructions to get the results of one calculation in position for the next.) ("car" is a mnemonic for "contents of the address part of register.")

cdr. *cdr* is handled in the same way as *car*, except that the result appears in the decrement part of the accumulator ("cdr" stands for "contents of the decrement part of register.")

cons. The value of $cons[x; y]$ must be the location of a register that has x and y in its address and decrement parts, respectively. There may not be such a register in the computer and, even if there were, it would be time-consuming to find it. Actually, what we do is to take the first available register from the *free-storage list*, put x and y in the address and decrement parts, respectively, and make the value of the function the location of the register taken. ("cons" is an abbreviation for "construct.")

It is the subroutine for *cons* that initiates the reclamation when the *free-storage list* is exhausted. In the version of the system that is used at present *cons* is represented by a closed subroutine. In the compiled version, *cons* is open.

e. *Representation of S-Functions by Programs.* The compilation of functions that are compositions of *car*, *cdr*, and *cons*, either by hand or by a compiler program, is straightforward. Conditional expressions give no trouble except that they must be so compiled that only the p's and e's that are required are computed. However, problems arise in the compilation of recursive functions.

In general (we shall discuss an exception), the routine for a recursive function uses itself as a subroutine. For example, the program for $subst[x; y; z]$ uses itself as a subroutine to evaluate the result of substituting into the subexpressions $car[z]$ and $cdr[z]$. While $subst[x; y; cdr[z]]$ is being evaluated, the result of the previous evaluation of $subst[x; y; car[z]]$ must be saved in a temporary storage register. However, *subst* may need the same register for evaluating $subst[x; y; cdr[z]]$. This possible conflict is resolved by the SAVE and UNSAVE routines that use the *public push-down list* [8]. The SAVE routine is entered at the beginning of the routine for the recursive function with a request to save a given set of consecutive registers. A block of registers called the *public push-down list* is reserved for this purpose. The SAVE routine has an index that tells it how many registers in the push-down list are already in use. It moves the contents of the registers which are to be saved to the first unused registers in the push-down list, advances the index of the list, and returns to the program from which control came. This program may then freely use these registers for temporary storage. Before the routine exits it uses UNSAVE, which restores the contents of the temporary registers from the push-down list and moves back the index of this list. The result of these conventions is described, in programming terminology, by saying that the recursive subroutine is transparent to the temporary storage registers.

f. *Status of the LISP Programming System* (February 1960). A variant of the function apply described in section 5f has been translated into a program APPLY for the IBM 704. Since this routine can compute values of S-functions given their descriptions as S-expressions and their arguments, it serves as an interpreter for the LISP programming language which describes computation processes in this way.

The program APPLY has been imbedded in the LISP programming system which has the following features:

1. The programmer may define any number of S-functions by S-expressions. these functions may refer to each other or to certain S-functions represented by machine language program.
2. The values of defined functions may be computed.
3. S-expressions may be read and printed (directly or via magnetic tape).

[8] 1995: now called a stack

4. Some error diagnostic and selective tracing facilities are included.

5. The programmer may have selected S-functions compiled into machine language programs put into the core memory. Values of compiled functions are computed about 60 times as fast as they would if interpreted. Compilation is fast enough so that it is not necessary to punch compiled program for future use.

6. A "program feature" allows programs containing assignment and **go to** statements in the style of ALGOL.

7. Computation with floating point numbers is possible in the system, but this is inefficient.

8. A programmer's manual is being prepared. The LISP programming system is appropriate for computations where the data can conveniently be represented as symbolic expressions allowing expressions of the same kind as subexpressions. A version of the system for the IBM 709 is being prepared.

5 Another Formalism for Functions of Symbolic Expressions

There are a number of ways of defining functions of symbolic expressions which are quite similar to the system we have adopted. Each of them involves three basic functions, conditional expressions, and recursive function definitions, but the class of expressions corresponding to S-expressions is different, and so are the precise definitions of the functions. We shall describe one of these variants called linear LISP.

The L-expressions are defined as follows:

1. A finite list of characters is admitted.

2. Any string of admitted characters in an L-expression. This includes the null string denoted by Λ.

There are three functions of strings:

1. $first[x]$ is the first character of the string x.

$first[\Lambda]$ is undefined. For example: $first[ABC] = A$

2. $rest[x]$ is the string of characters which remains when the first character of the string is deleted.

$rest[\Lambda]$ is undefined. For example: $rest[ABC] = BC$.

3. $combine[x; y]$ is the string formed by prefixing the character x to the string y. For example: $combine[A; BC] = ABC$

There are three predicates on strings:

1. $char[x]$, x is a single character.
2. $null[x]$, x is the null string.
3. $x = y$, defined for x and y characters.

The advantage of linear LISP is that no characters are given special roles, as are parentheses, dots, and commas in LISP. This permits computations with all expressions that can be written linearly. The disadvantage of linear LISP is that the extraction of subexpressions is a fairly involved, rather than an elementary, operation. It is not hard to write, in linear LISP, functions that correspond to the basic functions of LISP, so that, mathematically, linear LISP includes LISP. This turns out to be the most convenient way of programming, in linear LISP, the more complicated manipulations. However, if the functions are to be represented by computer routines, LISP is essentially faster.

6 Flowcharts and Recursion

Since both the usual form of computer program and recursive function definitions are universal computationally, it is interesting to display the relation between them. The translation of recursive symbolic functions into computer programs was the subject of the rest of this report. In this section we show how to go the other way, at least in principle.

The state of the machine at any time during a computation is given by the values of a number of variables. Let these variables be combined into a vector ξ. Consider a program block with one entrance and one exit. It defines and is essentially defined by a certain function f that takes one machine configuration into another, that is, f has the form $\xi' = f(\xi)$. Let us call f the associated function of the program block. Now let a number of such blocks be combined into a program by decision elements π that decide after each block is completed which block will be entered next. Nevertheless, let the whole program still have one entrance and one exit.

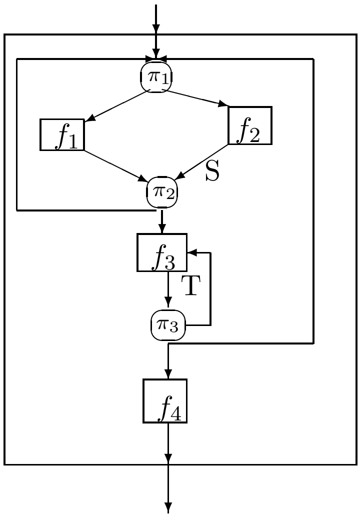


Figure 5

We give as an example the flowcart of figure 5. Let us describe the function $r[\xi]$ that gives the transformation of the vector ξ between entrance and exit of the whole block. We shall define it in conjunction with the functions s(ξ), and t[ξ], which give the transformations that ξ undergoes between the points S and T, respectively, and the exit. We have

$$
\begin{aligned}
r[\xi] &= [\pi_1 1[\xi] \to S[f_1[\xi]]; T \to S[f_2[\xi]]] \\
S[\xi] &= [\pi_2 1[\xi] \to r[\xi]; T \to t[f_3[\xi]]] \\
t[\xi] &= [\pi 3 I[\xi] \to f_4[\xi]; \pi_3 2[\xi] \to r[\xi]; T \to t[f_3[\xi]]]
\end{aligned}
$$

Given a flowchart with a single entrance and a single exit, it is easy to write down the recursive function that gives the transformation of the state vector from entrance to exit in terms of the corresponding functions for the computation blocks and the predicates of the branch. In general, we proceed as follows.

In figure 6, let β be an n-way branch point, and let $f_1, \cdots, f_n$ be the computations leading to branch points $\beta_1, \beta_2, \cdots, \beta_n$. Let ϕ be the function that transforms ξ between β and the exit of the chart, and let $\phi_1, \cdots, \phi_n$ be the corresponding functions for $\beta_1, \cdots, \beta_n$. We then write

$$\phi[\xi] = [p_1[\xi] \rightarrow \phi_1[f_1[\xi]]; \cdots; p_n[\xi] \rightarrow \phi_n[\xi]]]$$

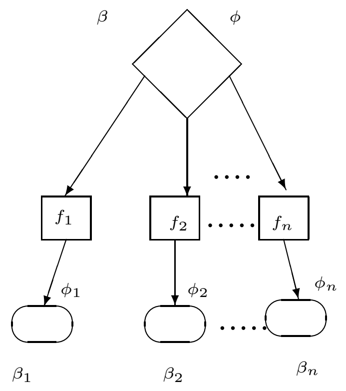


Figure 6

7 Acknowledgments

The inadequacy of the λ-notation for naming recursive functions was noticed by N. Rochester, and he discovered an alternative to the solution involving *label* which has been used here. The form of subroutine for *cons* which permits its composition with other functions was invented, in connection with another programming system, by C. Gerberick and H. L. Gelernter, of IBM Corporation. The LISP programming system was developed by a group including R. Brayton, D. Edwards, P. Fox, L. Hodes, D. Luckham, K. Maling, J. McCarthy, D. Park, S. Russell.

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