The Mutilated Checkerboard in Set Theory

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An 8 by 8 checkerboard with two diagonally opposite squares removed cannot be covered by dominoes each of which covers two rectilinearly adjacent squares. I present a set theory description of the proposition and an informal proof that the covering is impossible. While no present system that I know of will accept either formal description or the proof, I claim that both should be admitted in any heavy duty set theory.
We have the definitions

\[ \text{Board} = \mathbb{Z}_8 \times \mathbb{Z}_8, \]

\[ \text{mutilated-board} = \text{Board} - \{(0, 0), (7, 7)\}, \]

\[ \text{domino-on-board}(x) \equiv (x \subset \text{Board}) \land \text{card}(x) = 2 \]
\[ \land (\forall x_1 \ x_2)(x_1 \neq x_2 \land x_1 \in x \land x_2 \in x \]
\[ \supset \text{adjacent}(x_1, x_2), \]
\[\text{adjacent}(x_1, x_2) \equiv |c(x_1, 1) - c(x_2, 1)| = 1\]
\[\land c(x_1, 2) = c(x_2, 2)\]
\[\lor |c(x_1, 2) - c(x_2, 2)| = 1 \land c(x_1, 1) = c(x_2, 1),\]
and

\[\text{partial-covering}(z)\]
\[\equiv (\forall x)(x \in z \supset \text{domino-on-board}(x))\]
\[\land (\forall x \ y)(x \in z \land y \in z \supset x = y \lor x \cap y = \{\})\]
Theorem:
\[ \neg (\exists z) (\text{partial-covering}(z) \land \bigcup z = \text{mutilated-board}) \]

Proof:

We define

\[ x \in Board \ni \text{color}(x) = \text{rem}(c(x, 1) + c(x, 2), 2) \]
$domino$-$on$-$board(x) \supset (\exists u \ v)(u \in x \land v \in x \land color(u) = 0 \land color(v) = 1)$, 

$partial$-$covering(z) \supset \begin{align*}
\text{card} & (\{u \in \bigcup z | color(u) = 0\}) \\
& = \text{card} (\{u \in \bigcup z | color(u) = 1\}),
\end{align*}$

$\text{card}(\{u \in mutilated$-$board | color(u) = 0\})$

$\neq \text{card}(\{u \in mutilated$-$board | color(u) = 1\})$, 

and finally
\neg (\exists z)(\text{partial-covering}(z) \land \text{mutilated-board} = \bigcup z) \quad (11)

Q.E.D.