The Mutilated Checkerboard in Set Theory

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An 8 by 8 checkerboard with two diagonally opposite squares removed cannot be covered by dominoes each of which covers two rectilinearly adjacent squares. We present a set theory description of the proposition and an informal proof that the covering is impossible. While no present system that I know of will accept either the formal description or the proof, I claim that both should be admitted in any *heavy duty set theory*.

We have the definitions

$$Board = Z8 \times Z8, \tag{1}$$

$$mutilated\text{-}board = Board - \{(0,0),(7,7)\}, \tag{2}$$

$$\begin{aligned} &domino\text{-}on\text{-}board(x) \equiv (x \subset Board) \wedge card(x) = 2 \\ &\wedge(\forall x1\ x2)(x1 \neq x2 \wedge x1 \in x \wedge x2 \in x \\ &\supset adjacent(x1, x2)), \end{aligned} \tag{3}$$

$$
\begin{aligned}
&adjacent(x1, x2) \equiv |c(x1, 1) - c(x2, 1)| = 1 \\
&\wedge c(x1, 2) = c(x2, 2) \\
&\vee |c(x1, 2) - c(x2, 2)| = 1 \wedge c(x1, 1) = c(x2, 1),
\end{aligned}
\tag{4}
$$

and

$$
\begin{aligned}
&partial\text{-}covering(z) \\
&\equiv (\forall x)(x \in z \supset domino\text{-}on\text{-}board(x)) \\
&\wedge (\forall x\ y)(x \in z \wedge y \in z \supset x = y \vee x \cap y = \{\})
\end{aligned}
\tag{5}
$$

Theorem:

$$\neg(\exists z)(partial\text{-}covering(z) \wedge \bigcup z = mutilated\text{-}board) \quad (6)$$

Proof:

We define

$$x \in Board \supset color(x) = rem(c(x,1) + c(x,2), 2) \quad (7)$$

$$domino\text{-}on\text{-}board(x) \supset \\ (\exists u\ v)(u \in x \wedge v \in x \wedge color(u) = 0 \wedge color(v) = 1), \tag{8}$$

$$partial\text{-}covering(z) \supset \\ card(\{u \in \bigcup z | color(u) = 0\}) \\ = card(\{u \in \bigcup z | color(u) = 1\}), \tag{9}$$

$$card(\{u \in mutilated\text{-}board | color(u) = 0\}) \\ \neq card(\{u \in mutilated\text{-}board | color(u) = 1\}), \tag{10}$$

and finally

$$\neg(\exists z)(\mathit{partial\text{-}covering}(z) \ \wedge \ \mathit{mutilated\text{-}board} = \bigcup z) \quad (11)$$

Q.E.D.