Examples of proofs using Kripke structures:
(taken from Fagin, Halpern, Moses, and Vardi: *Reasoning about Knowledge*, MIT Press, 1995)

1. Muddy children puzzle:

Assumptions: Assume 3 children (actually works for any n, but easier to draw for n=3), who are perfect reasoners, are always truthful, and always given an answer if they know it. . The children are playing outside, and may get muddy foreheads. Any child can see if the other children have muddy forheads but can’t see his or her own forehead. At one point, an adult says to them: at least one of you has mud on your forehead.

The adult now says: Do any of you know if you have mud on your forehead?

No one answers. The adult repeats the question, and again no one answers.

The third time the adult asks the question, one or more of the children answer.

The question:

How many children have muddy foreheads?

Answer: 3 children.

In general, it can be shown that if n children have muddy foreheads, they will all answer on the nth round.

We will show this using Kripke structures for the case of n=3.

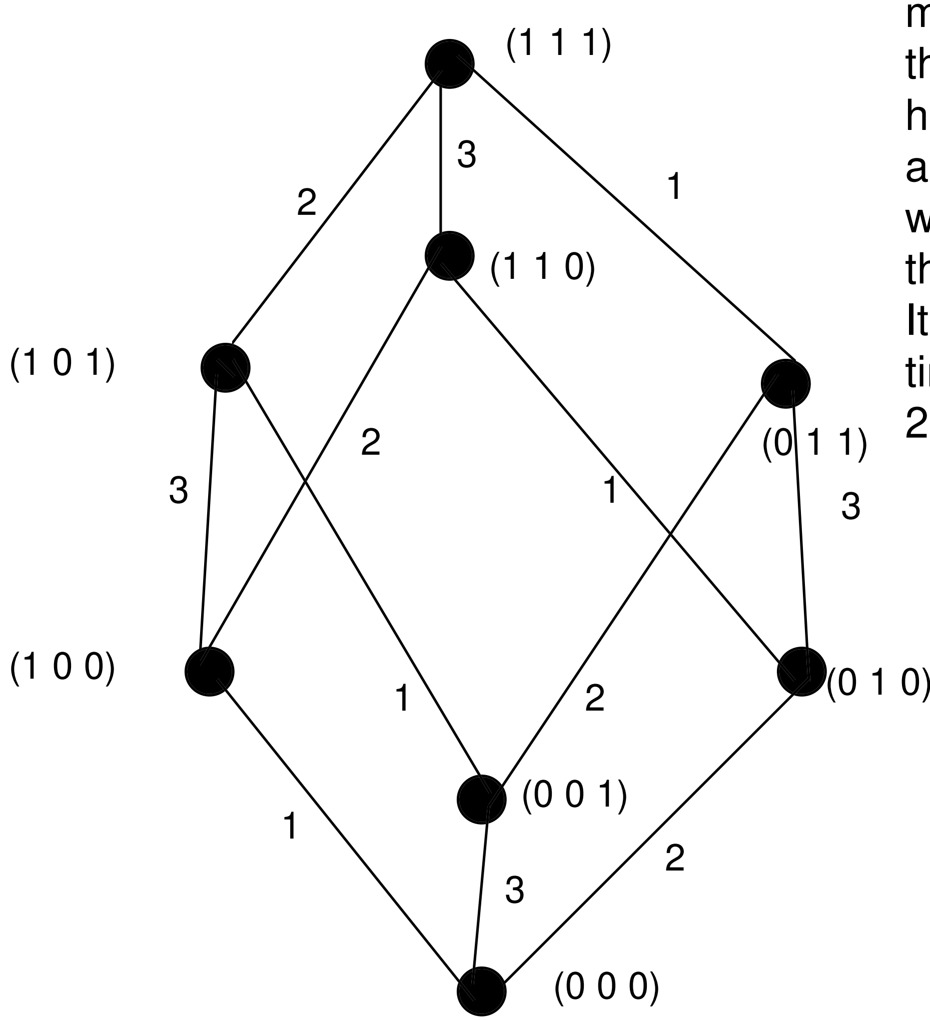
How to represent the muddy children puzzle using Kripke structures:

1. **Decide on the representation of a state; Enumerate all possible states**.
(Each state is a possible world in the Kripke structure.)
There are 3 children, and each child may be muddy or not muddy.
Thus there are 2**3 = 8 possible states.
We can represent a state as a tuple of 3 booleans, where a 1 in the nth position means that the nth child is muddy, and a 0 in the nth position means that the nth child is not muddy.
For example (110) represents the state where children 1 and 2 are muddy and child 3 is not muddy.
The states are:
(111) (110) (101) (011) (100) (001) (010) (000).

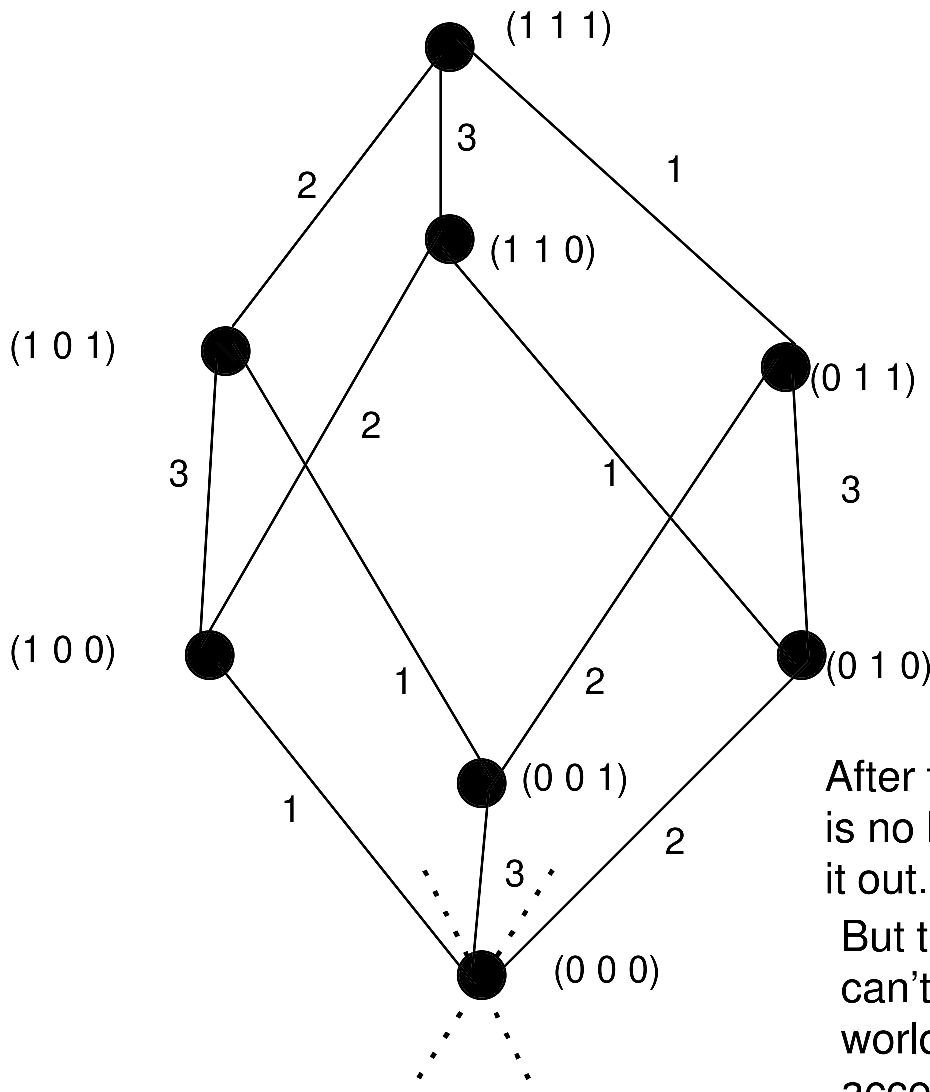
2. **Arrange the states in a Kripke structure. That is, determine which states are accessible from others from the different agent's points of view.**
For example, consider the state (111). Since child 1 does not know whether or not he or she is muddy, as far as child 1 is concerned, he or she could be in statte (011). However, (011) is not accessible from (111) for child 2, since child 2 knows in (111) that child 1 is muddy.

3. **Determine how the knowledge structure changes after each action (in this case, act of communication).**
Draw the structure resulting after each action.

4. **Stop when only one world is left.**
(For general knowledge puzzles, you may not be able to get down to 1 world; go as far as you can.)



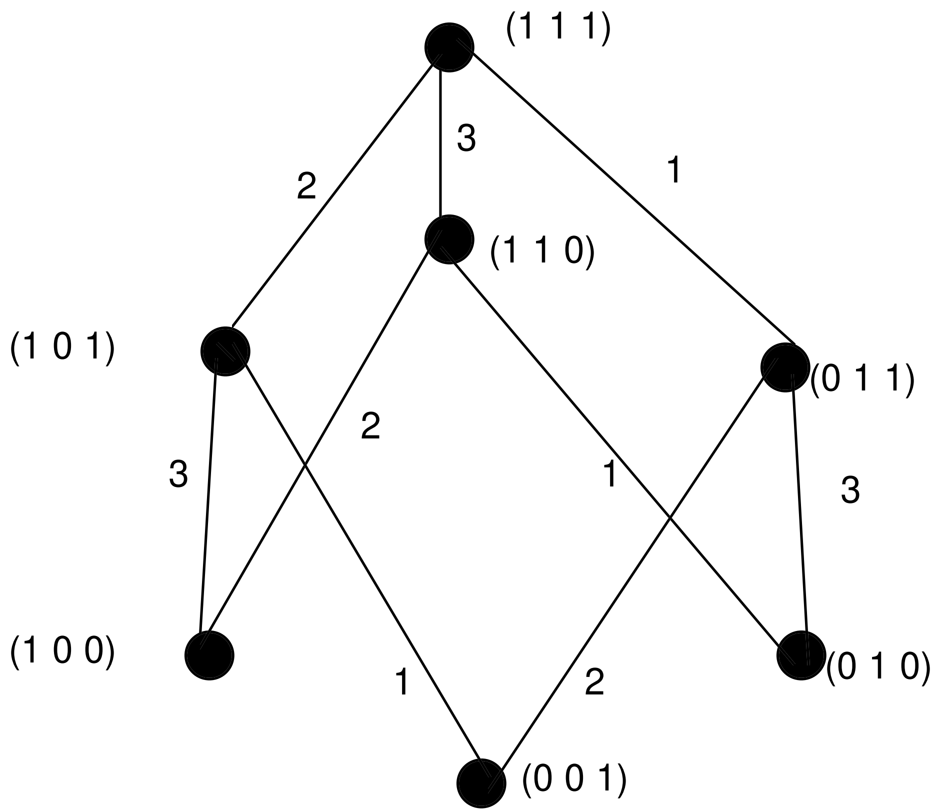
This is the Kripke structure for the muddy children puzzle before the adult speaks. It shows how all possible states (worlds) are related. It says nothing about which of these possible worlds is the real state.
It's important to notice that at any time, an agent only considers 2 worlds possible.



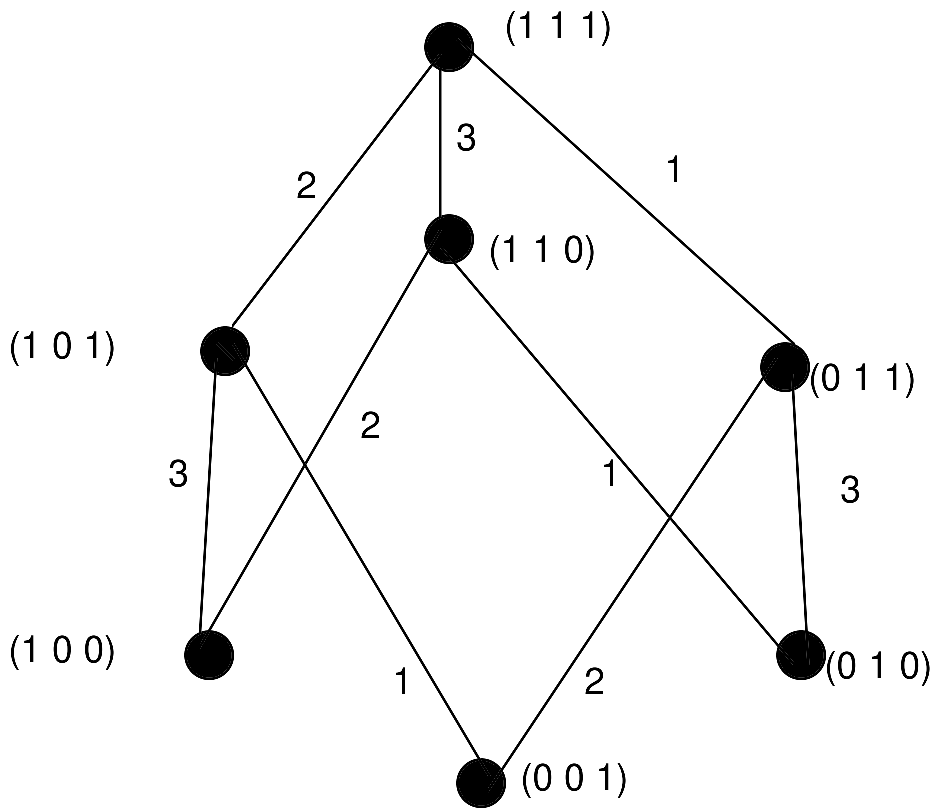
After the adult speaks, the bottom state is no longer possible. So we can cross it out.

But that also means that this world can't be accessible from any other world. So we can cross out the accessibility lines.

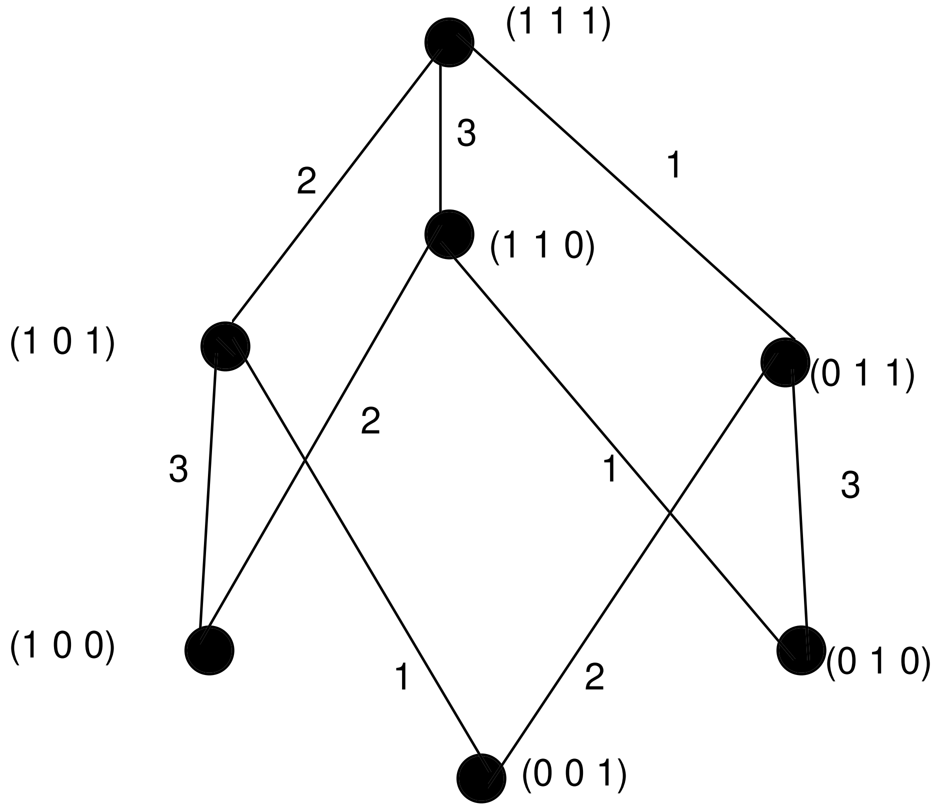
This gives us ....



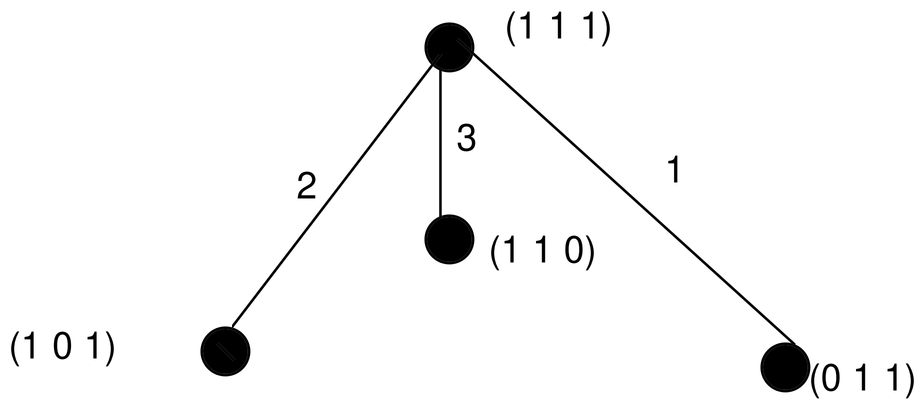
This is the Kripke structure after the adult speaks.
It is no longer the case that (0 0 0) is a possible world.

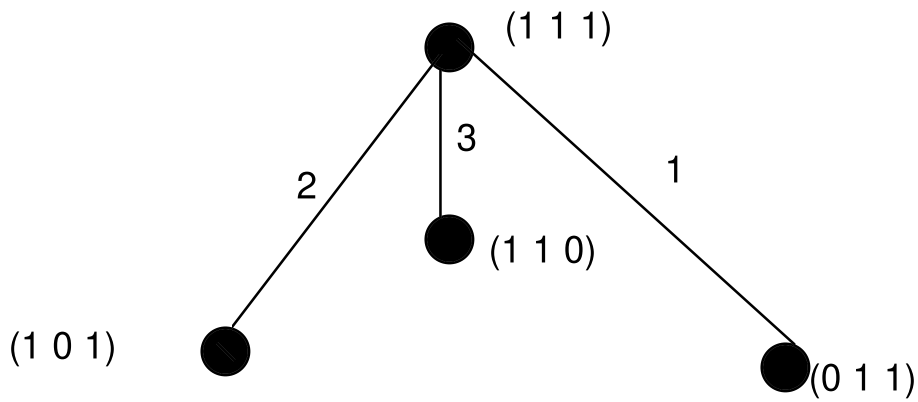


Now the adult asks the children if anyone knows if he or she is muddy.
Now, assume that (1 0 0) were the real state of the world. Prior to the adult's statement, child 1 could have thought that (0 0 0) was a possible world. Since now that world is impossible, only (1 0 0) is possible; that is, child 1 knows that he or she is muddy.
Thus, if (1 0 0) were the real world, child 1 would speak. By similar arguments, we can show that if (0 1 0) were the real world, child 2 would speak, and if (0 0 1) were the real world, child 3 would speak.
Since none of the children speak, we know these worlds are not the real worlds. Thus, ...

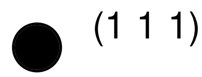


Thus, if no one speaks after the adult first asks the question,
we can eliminate the impossible worlds, and the lines going into them.
This gives ....


(1 1 1)
3
2
1
(1 1 0)
(1 0 1)
(0 1 1)



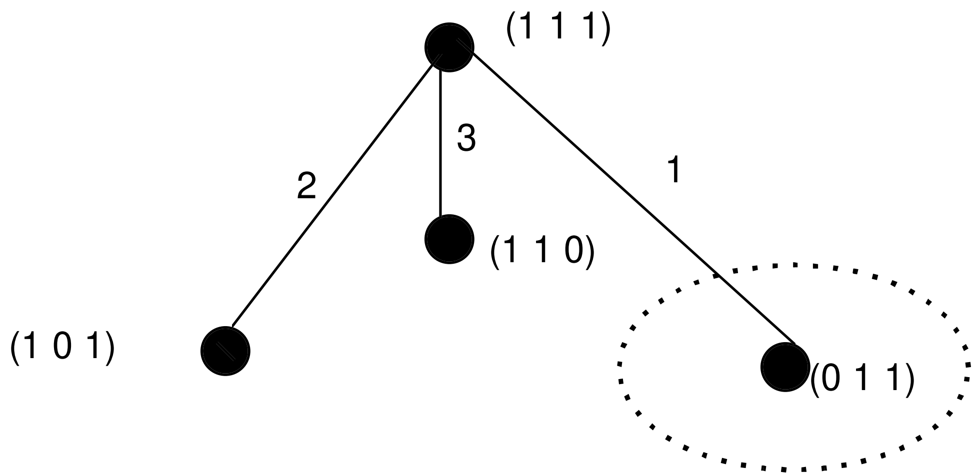
Again, the adult asks the question. Again, the children can't answer because they are not sure whether there are two or three muddy children.
Now, consider the situation from child 1's point of view. He or she knows that if the real world were (1 1 0), child 2 would have originally thought both (1 1 0) and (1 0 0) were possible worlds. Since (1 0 0) was eliminated in the last round of questioning, if (1 1 0) were the real world, child 2 could now answer ...
Similar arguments cand be made for children 2 and 3.
Thus, after the second round of questioning and silence, the structure changes once again, and becomes ...



After the 2nd round of questioning, this is the only world left.
All children know they're muddy. So at the next round of questioning, they all say that they know they are dirty.

Note: This is the sort of argument expected for problem 2.

What would have happened if there had been only 2 dirty children? Let's assume, without loss of generality, that children 2 and 3 are dirty. As before the structure would have changed when the adult told them that at least one child was muddy. Furthermore, no one would have been able to answer when the adult first asked if anyone knew if he or she was muddy, and the structure would have changed again. But in this case .....



In this world, only child 1 thinks another world is possible: the world where he or she might be muddy. On the other hand, children 2 and 3 each know that it is not the case that only 1 child is dirty; since that was their only other possibility, they know know that 2 children are dirty.
When the adult asks, they say so.