Completeness of a Gentzen System for First Order Logic

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0.1 First Order Logic

A first order language L is a a set of function symbols, predicate symbols and constant symbols. Each function and predicate symbol has a positive number associated with it, its arity $\#(f) = n$.

Given a langauge L we have a notion of a structure for L.

A *structure* for L is a pair $\mathfrak{M} = \langle M, F \rangle$ where M is a non-empty set and F is a operation on the domain L such that

- if $P \in L$ is a n-ary predicate symbol the $R^{\mathfrak{M}} \subseteq M^n$;
- if $f \in L$ is a n-ary function symbol the $f^{\mathfrak{M}}: M^n \to M$;
- if $c \in L$ is a constant symbol the $c^{\mathfrak{M}} \in M$;

One often write $\mathfrak{M}$ as $\langle M, R^{\mathfrak{M}}, \ldots, f^{\mathfrak{M}}, \ldots, c^{\mathfrak{M}} \rangle$.

Definition 1 *The* terms *of L form the smallest set of expressions containing the variables, $x, y, z, \ldots,$, all constant symbols in L and cosed under the formation rule: if $t_1, \ldots, t_n$ are terms of L and if $f \in L$ is an n-ary function symbol then the expression $f(t_1, \ldots, t_n)$ is a term of L. A* closed term *is a term in which no variable appears.*

Definition 2 *An atomic formula of L is an expression of either of then two forms:*

$$(t_1 = t_2) \qquad P(t_1, \ldots, t_n)$$

where in the first case t_1 and t_2 are terms of L. IN the second case $P \in L$ is any n-ary predicate symbol and $t_1, \ldots, t_n$ are terms of L.

Definition 3 *The first order* formulas *of L form the smallest set of expressions containing the atomic formulas and closed under the following formation rules:*

- *If ϕ, ψ are formulas so are the expressions.*

$$\neg\phi, \quad (\phi \vee \psi) \quad (\phi \wedge \psi) \quad (\phi \supset \psi)$$

- *if ϕ is a formula and v is a variable, the $(\exists v\phi)$ and $\forall v\phi)$ are formulas.*

Definition 4 *The set of $FV(\phi)$* free variables *of a formula ϕ is defined as follows:*

- *if ϕ is an atomic formula, the $FV(\phi)$ is just the set of variables appearing in the expression ϕ.*
- $FV(\neg\phi) = FV(\phi)$
- $FV(\phi \vee \psi) = FV(\phi \wedge \psi) = FV(\phi \supset \psi) = FV(\phi) \cup FV(\psi)$
- $FV(\exists v\phi) = FV(\forall v\phi) = FV(\phi) - \{v\}$

Definition 5 *A first order* sentence *is a fomula without any free variables.*

Definition 6 *Let $\mathfrak{M} = \langle M, \dots \rangle$ be a structure for a language L. An* assignment *in $\mathfrak{M}$ is a function s with domain the set of variables of L and range a subset of M. We think of s as assigning a meaning $s(v)$ to the variables v. We can then define for each term t of L, a function $t^{\mathfrak{M}}$ which maps assignments to elements of M.*

Let M be given. For a term t of L we define $t^{\mathfrak{M}}$ as follows:

- *If t is a constant symbol c then $t^{\mathfrak{M}}(s) = c^{\mathfrak{M}}$ for all s.*
- *if t is a variable v, then $t^{\mathfrak{M}}(s) = s(v)$ for all s,*
- *if t is a term $f(t_1, \dots, t_n)$ then forall s define*

$$t^{\mathfrak{M}} = f^{\mathfrak{M}}(t_1^{\mathfrak{M}}(s), \dots, t_n^{\mathfrak{M}}(s)).$$

Definition 7 *In the following definition we use* $s(\begin{smallmatrix} a \\ c \end{smallmatrix})$ *for the assignment* s' *which agrees with* s *except that* $s'(v) = a$.

Let $\mathfrak{M}$ *be an* L *structure. We define a relation*

$$\mathfrak{M} \models \phi[s]$$

(read the assignment s *satisfies the formula* ϕ *in* $\mathfrak{M}$*) for all assignments* s *and all formulas* ϕ *as follows:*

- $\mathfrak{M} \models (t_1 = t_2)[s]$ *iff* $t_1^{\mathfrak{M}}(s) = t_2^{\mathfrak{M}}(s)$,
- $\mathfrak{M} \models P(t_1, \ldots, t_n)[s]$ *iff* $(t_1^{\mathfrak{M}}(s), \ldots, t_n^{\mathfrak{M}}(s)) \in P^{\mathfrak{M}}$,
- $\mathfrak{M} \models \neg\phi[s]$ *iff not* $\mathfrak{M} \models \phi[s]$,
- $\mathfrak{M} \models (\phi \vee \psi)[s]$ *iff* $\mathfrak{M} \models \phi[s]$ *or* $\mathfrak{M} \models \psi[s]$,
- $\mathfrak{M} \models (\phi \wedge \psi)[s]$ *iff* $\mathfrak{M} \models \phi[s]$ *and* $\mathfrak{M} \models \psi[s]$,
- $\mathfrak{M} \models (\phi \supset \psi)[s]$ *iff either not* $\mathfrak{M} \models \phi[s]$ *or else* $\mathfrak{M} \models \psi[s]$,
- $\mathfrak{M} \models (\forall v\phi)[s]$ *iff for all* $a \in M$, $\mathfrak{M} \models \phi[s(\begin{smallmatrix} a \\ c \end{smallmatrix})]$,
- $\mathfrak{M} \models (\forall v\phi)[s]$ *iff there exists an* $a \in M$, *such that* $\mathfrak{M} \models \phi[s(\begin{smallmatrix} a \\ c \end{smallmatrix})]$.

Definition 8 *When* ϕ *is a sentence, (i.e. has no free variables), we write* $\mathfrak{M} \models \phi$ *for* $\mathfrak{M} \models \phi[s]$ *as the truth of this does no depend on* s.

A structure $\mathfrak{M}$ *is a* model of a *set of sentences* Φ *if* $\mathfrak{M} \models \phi$ *for all* $\phi \in \Phi$.

1 Proof Theory

First order logic has as connectives $\wedge$, $\vee$, $\supset$ and $\sim$. It can be axiomatized in a Gentzen system as follows. Gentzen System Sequents $\Gamma \Rightarrow \Delta$ should be read $\wedge\Gamma \supset \vee\Delta$, where $\Gamma \cup \Delta$ is a finite set of formulas.

Axioms

$$\Gamma, \varphi \Rightarrow \Delta, \varphi$$

$$\Gamma, \varphi \Rightarrow \Delta, t = t (t \text{ any term})$$

Rules of Inference

$$\Rightarrow \wedge \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \qquad \wedge \Rightarrow \quad \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$$

$$\vee \Rightarrow \quad \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \qquad \Rightarrow \vee \quad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$\supset\Rightarrow \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \supset \psi, \Gamma \Rightarrow \Delta} \qquad \Rightarrow\supset \quad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$$

$$\sim\Rightarrow \quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\sim \varphi, \Gamma \Rightarrow \Delta} \qquad \Rightarrow\sim \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim \varphi}$$

if E is $t_1 = t_2$ or $t_2 = t_1$, then

$$\frac{\Gamma, \phi(t_1) \Rightarrow \Delta, \psi(t_1)}{\Gamma, E, \phi(t_2) \Rightarrow \Delta, \psi(t_2)}$$

$$\forall \Rightarrow \quad \frac{\Gamma, \phi(t) \Rightarrow \Delta}{\Gamma, \forall v \phi(v) \Rightarrow \Delta} \quad \Rightarrow \exists \quad \frac{\Gamma \Rightarrow \Delta, \phi(t)}{\Gamma \Rightarrow \Delta, \exists v \phi(v)}$$

For the next two rules the variable v is not allowed occur free in $\Gamma \cup \Delta$.

$$\Rightarrow \forall \quad \frac{\Gamma \Rightarrow \Delta \phi(v)}{\Gamma \Rightarrow \Delta, \forall y \phi(y)} \quad \exists \Rightarrow \quad \frac{\Gamma, \phi(v) \Rightarrow \Delta}{\Gamma, \exists y \phi(y) \Rightarrow \Delta}$$

$$\text{Cut} \quad \frac{\Gamma, \phi \Rightarrow \Delta \quad \Gamma \Delta, \phi}{\Gamma \Rightarrow \Delta}$$

2 Completeness Theorem

A sentence ϕ is derivable from a set of sentences T in the system G iff $T \models \phi$.

The $\Rightarrow$ direction follows from the fact that each rules preserves validity, and the fact that the axioms are valid.

Definition 9 *A* theory *is a set of sentences with the property that $T \vdash \phi \Rightarrow \phi \in T$ (a theory is closed under derivability.*

Definition 10 *A set Γ such that $T = \{phi \Gamma \vdash \phi\}$ is called an axiom set of the theory T. The elements if Γ are called axioms.*

Definition 11 *T is called a* Henkin Theory *if for every sentence $\exists v.\phi(v)$, there is a constant c such that $\exists v.\phi(v) \to \phi(c) \in T$ (such a c is called a* witness *for v in ϕ).*

Definition 12 *T' is an extension of T if $T \subseteq T'$.*

T' is a conservative extension of T if $T' \cap L = T$ (i.e. all theorems of T' in the language L are already theorems of T.

Definition 13 *Let T be a theory with language L. We add to the language for each sentence δ of the form $\exists v\phi(v)$, a constant c_δ such that distinct δ's yield distinct c_δs.. The resulting language is L^*.*

T^* is the theory with axiom set $T \cup \{\exists x\phi(x) \supset \phi(c_\delta) | \exists v\phi(v)$ closed , with witness $c_\delta\}$.

Lemma 14 *T^* is conservative over T.*

Let $\exists x\phi(x) \supset \phi(c)$ be one of the new axioms. Suppose $\Gamma, \exists x\phi(x) \supset \phi(c) \vdash \psi$ where ψ does not contain c and where Γ is a set of sentences none of which contain c. We show that $\Gamma \vdash \psi$.

$\Gamma \Rightarrow (\exists x\phi(x) \supset \phi(c) \supset \psi$

$\Gamma \Rightarrow (\exists x\phi(x) \supset \phi(y) \supset \psi$ where y is a variable that does not occu in the derivation of the previous line, as we can replace all occurences of c by y, and the proof will still be valid.

$\Gamma, \exists x\phi(x) \supset \phi(y) \Rightarrow \psi$

$\Gamma, \exists x\phi(x) \Rightarrow \phi(y), \psi$

$\Gamma, \exists x\phi(x) \Rightarrow \exists y\phi(y), \psi$ as y does not occur in Γ, ψ.

$\Gamma, \exists x\phi(x) \supset \exists y\phi(y) \Rightarrow \psi$

But we have $\Rightarrow \exists x\phi(x) \supset \exists y\phi(y)$ so $\Gamma \Rightarrow \exists x\phi(x) \supset \exists y\phi(y)$.

$\Gamma \Rightarrow \psi$ by Cut.

Now let $T^* \vdash \psi$ for a $\psi \in L$. By the definition of derivability there is a finite set of axioms from T^* that are needed to derive ψ. These are either sentences from T or of the form $\exists x\phi(x) \supset \phi(c)$. Therefore $T \cup \{\delta_1, \ldots, \delta_n\} \vdash \psi$.

We show $T \vdash \psi$ by induction. For $n = 0$ we are done. For the induction step we use the above.

Lemma 15 *Define $T_0 = T; T_{n+1} = (T_n)^*$; then $T_\omega = \cup\{T_n | n \geq 0\}$ is a Henkin theory and T_ω is conservative over T.*

Call the language of T_n L_n.

T_ω is a theory: We need to show that $T_\omega \vdash \delta$, if $\psi_0, \ldots \psi_n \vdash \delta$ for certain ψ_i in T_ω.

For each $i \leq n$ $\psi_i \in T_{m_i}$ for some m_i.

Let $m = max\{m_i | i \leq n\}$. Since $T_l \subseteq T_{k+1}$ for all k, we have $T_{m_i} \subseteq T_m (i \leq n)$

Therefore $T_m \vdash \delta$.

T_m is by definition a theory, so $\delta \in T_m \subseteq T_\omega$.

T_ω is a Henkin theory. Le $\exists x \phi(x) \in L_\omega$, then $\exists x \phi(x) \in L_n$ for some n. By definition $\exists \phi(x) \supset \phi(c) \in T_{n+1}$ for a certain c. So $\exists x \phi(x) \supset \phi(x) \in T_\omega$.

T_ω is conservative over T. Observer that $T_\omega \vdash \delta$ if $\mathcal{T}_n \vdash \delta$ for some n. By induction of n, we can establish that $T_\omega \vdash \delta$ if $\mathcal{T} \vdash \delta$.

Lemma 16 Zorn's Lemma: *Every partially ordered set in which every chain (i.e. totally ordered subset) has an upper bound contains at least one maximal element.*

Lemma 17 *Each consistent theory is contained in a maximally consistent theory.*

Consider the set A of conservative extensions of T ordered by inclusion. Each chain has an upper bound, as the union of the chain is a conservative extension of T.

Therefore A has a maximal element T_m.

T_m is a maximally conservative extension of T. $T_m \subseteq T'$ and $T' \in A$ then $T_m = T$.

Lemma 18 *If T_m is a maximally consistent extension of T_ω, then T_m is a Henkin theory.*

Being a Henkin Theory is presevered under taking a maximally consistent extension as the language remains fixed.

Theorem 19 *If Γ is consistent, then Γ has a model.*

Let $T = \{\delta | \Gamma \vdash \delta\}$ be the theory of Γ. Any model of T is a model of Γ.

Let T_m be a maximally consistent extension of T_ω, with language L_m.

We construct a model of T_m from T_m itself.

The universe S of the model is the closed terms of T_m.

For each function symbol f we define a function $\overline{f}\colon A^k \to A = f(t_1 \ldots t_k)$.

For each predicet symbol P we define a relation $\overline{P}\colon A^k$ by $(t_1 \ldots t_k) \in \overline{P}$ iff $T_m \vdash P(t_1, \ldots, t_k)$.

For each contant symbol c we define $\overline{c} = c$.

This is not a model as equality is not interpreted by real equality.

We create the real model by taking the quotient of this model by $t \sim s$, where $t \sim s$iff $t \sim s \in T_m$.

We need to show that $t \sim s$ is an equivalence relation, i.e. symetric, transitive and reflexive.

We need to show that all predicates and functions are congruent on $\sim$.

This follows from the axioms for equality.

We call this model $\mathfrak{A}$.

We now show that if $\mathfrak{A} \models \phi$ then $T_m \vdash \phi$ by induction on ϕ

The base cases are obvious by the definition of the model.

The induction step for propositional connectives follow from the right elimiation rules and cut.

The induction step for $\forall$ is the following.

Given $\mathfrak{A} \models \phi(c)$ for all $c \in A$ iff $T_m \vdash \phi(c)$ for all c in A, show

n $\mathfrak{A} \models \forall\phi(v)$ iff $T_m \vdash \forall\phi(v)$.

We assume the left, and try to derive the right.

$\mathfrak{A} \models \forall\phi(v)$. Consider the Henkin sentence $\exists x.\neg\phi(x) \supset \neg phi(c)$.

This is equivalent to $\phi(c) \supset \forall x.\phi(x)$. We have that $\mathfrak{A} \models \phi(c)$ iff $T_m \vdash \phi(c)$, so by one application of $\supset\Rightarrow$ and cut we can derive that $T_m \vdash \forall x.\phi(x)$ as required.

The opposite direction follows from $\forall \Rightarrow$ and cut by instantiating with the c in question.

Theorem 20

$$\Gamma \vdash \phi \textit{ iff } \Gamma \models \phi$$

The left to right part is Soundness. We now consider

$$\Gamma \models \phi \text{ implies } \Gamma \vdash \phi$$

We consider the contrapositive:

$$\Gamma \nvdash \phi \text{ implies } \Gamma \not\models \phi$$

$\Gamma \vdash \phi$ is equivalent to $\Gamma, \neg\phi$ is consistent.

We use the earlier theorem to create a structure $\mathfrak{M}$ where $\mathfrak{M} \models \Gamma, \neg\phi$.

By construction , $\mathfrak{M} \models \Gamma$ and $\mathfrak{M} \models \neg\phi$. Therefore $\mathfrak{M} \not\models \phi$. Therefore we have a model of Γ which is not a model of ϕ as required.