# Completeness of a Gentzen System for First Order Logic

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A first order language $L$ is a a set of function symbols, predicate symbols and constant symbols. Each function and predicate symbol has a positive number associated with it, its arity $\#(f) = n$.

Given a language $L$ we have a notion of a structure for $L$.

A *structure* for $L$ is a pair $\mathfrak{M} = \langle M, F \rangle$ where $M$ is a non-empty set and $F$ is a operation on the domain $L$ such that

- if $P \in L$ is a $n$-ary predicate symbol the $R^{\mathfrak{M}} \subseteq M^n$;
- if $f \in L$ is a $n$-ary function symbol the $f^{\mathfrak{M}} \colon M^n \to M$;
- if $c \in L$ is a constant symbol the $c^{\mathfrak{M}} \in M$;

One often write $\mathfrak{M}$ as $\langle M, R^{\mathfrak{M}}, \dots, f^{\mathfrak{M}}, \dots, c^{\mathfrak{M}} \rangle$.

**Definition 1** *The* terms *of $L$ form the smallest set of expressions containing the variables, $x, y, z, \dots,$, all constant symbols in $L$ and closed under the formation rule: if $t_1, \dots, t_n$ are terms of $L$ and if $f \in L$ is an $n$-ary function symbol then the expression $f(t_1, \dots, t_n)$ is a term of $L$. A* closed term *is a term in which no variable appears.*

**Definition 2** *An atomic formula of $L$ is an expression of either of then two forms:*

$$(t_1 = t_2) \qquad P(t_1, \dots, t_n)$$

*where in the first case $t_1$ and $t_2$ are terms of $L$. IN the second case $P \in L$ is any $n$-ary predicate symbol and $t_1, \dots, t_n$ are terms of $L$.*

**Definition 3** *The first order* formulas *of $L$ form the smallest set of expressions containing the* *atomic formulas and closed under the following formation rules:*

- *If $\phi, \psi$ are formulas so are the expressions.*

$$\neg\phi, \quad (\phi \vee \psi) \quad (\phi \wedge \psi) \quad (\phi \supset \psi)$$

- *if $\phi$ is a formula and $v$ is a variable, the $(\exists v \phi)$ and $\forall v \phi)$ are formulas.*

**Definition 4** *The set of $FV(\phi)$* free variables *of a formula $\phi$ is defined as follows:*

- *if $\phi$ is an atomic formula, the $FV(\phi)$ is just the set of variables appearing in the expression $\phi$.*

- $FV(\neg\phi) = FV(\phi)$

- $FV(\phi \vee \psi) = FV(\phi \wedge \psi) = FV(\phi \supset \psi) = FV(\phi) \cup FV(\psi)$

- $FV(\exists v\phi) = FV(\forall v\phi) = FV(\phi) - \{v\}$

**Definition 5** *A first order* sentence *is a formula without any free variables.*

**Definition 6** *Let $\mathfrak{M} = \langle M, \ldots \rangle$ be a structure for a language $L$. An* assignment *in $\mathfrak{M}$ is a function $s$ with domain the set of variables of $L$ and range a subset of $M$. We think of $s$ as assigning a meaning $s(v)$ to the variables $v$. We can then define for each term $t$ of $L$, a function $t^{\mathfrak{M}}$ which maps assignments to elements of $M$.*

*Let $M$ be given. For a term $t$ of $L$ we define $t^{\mathfrak{M}}$ as follows:*

- *If $t$ is a constant symbol $c$ then $t^{\mathfrak{M}}(s) = c^{\mathfrak{M}}$ for all $s$.*

- *if $t$ is a variable $v$, then $t^{\mathfrak{M}}(s) = s(v)$ for all $s$,*

- *if* $t$ *is a term* $f(t_1, \ldots, t_n)$ *then forall* $s$ *define*

$$t^{\mathfrak{M}} = f^{\mathfrak{M}}(t_1^{\mathfrak{M}}(s), \ldots, t_n^{\mathfrak{M}}(s)).$$

**Definition 7** *In the following definition we use* $s(\begin{smallmatrix} a \\ c \end{smallmatrix})$ *for the assignment* $s'$ *which agrees with* $s$ *except that* $s'(v) = a$.

*Let* $\mathfrak{M}$ *be an* $L$ *structure. We define a relation*

$$\mathfrak{M} \models \phi[s]$$

*(read the assignment* $s$ *satisfies the formula* $\phi$ *in* $\mathfrak{M}$*) for all assignments* $s$ *and all formulas* $\phi$ *as follows:*

- $\mathfrak{M} \models (t_1 = t_2)[s]$ *iff* $t_1^{\mathfrak{M}}(s) = t_2^{\mathfrak{M}}(s)$,

- $\mathfrak{M} \models P(t_1, \ldots, t_n)[s]$ *iff* $(t_1^{\mathfrak{M}}(s), \ldots, t_n^{\mathfrak{M}}(s)) \in P^{\mathfrak{M}}$,

- $\mathfrak{M} \models \neg\phi[s]$ *iff not* $\mathfrak{M} \models \phi[s]$,

- $\mathfrak{M} \models (\phi \vee \psi)[s]$ *iff* $\mathfrak{M} \models \phi[s]$ *or* $\mathfrak{M} \models \psi[s]$,

- $\mathfrak{M} \models (\phi \wedge \psi)[s]$ *iff* $\mathfrak{M} \models \phi[s]$ *and* $\mathfrak{M} \models \psi[s]$,

- $\mathfrak{M} \models (\phi \supset \psi)[s]$ *iff either not* $\mathfrak{M} \models \phi[s]$ *or else* $\mathfrak{M} \models \psi[s]$,

- $\mathfrak{M} \models (\forall v\phi)[s]$ *iff for all* $a \in M$, $\mathfrak{M} \models \phi[s( \begin{smallmatrix} a \\ c \end{smallmatrix} )]$,

- $\mathfrak{M} \models (\forall v\phi)[s]$ *iff there exists an* $a \in M$, *such that* $\mathfrak{M} \models \phi[s( \begin{smallmatrix} a \\ c \end{smallmatrix} )]$.

**Definition 8** *When* $\phi$ *is a sentence, (i.e. has no free variables), we write* $\mathfrak{M} \models \phi$ *for* $\mathfrak{M} \models \phi[s]$ *as the truth of this does no depend on* $s$.

*A structure* $\mathfrak{M}$ *is a* model of a *set of sentences* $\Phi$ *if* $\mathfrak{M} \models \phi$ *for all* $\phi \in \Phi$.

First order logic has as connectives $\wedge$, $\vee$, $\supset$ and $\sim$. It can be axiomatized in a Gentzen system as follows. Gentzen System Sequents $\Gamma \Rightarrow \Delta$ should be read $\wedge\Gamma \supset \vee\Delta$, where $\Gamma \cup \Delta$ is a finite set of formulas.

## Axioms

$$\Gamma, \varphi \Rightarrow \Delta, \varphi$$

$$\Gamma, \varphi \Rightarrow \Delta, t = t (t \text{ any term})$$

## Rules of Inference

$$\Rightarrow \wedge \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \qquad \wedge \Rightarrow \quad \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$$

$$\vee \Rightarrow \quad \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \qquad \Rightarrow \vee \quad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$\supset\Rightarrow \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \supset \psi, \Gamma \Rightarrow \Delta} \qquad \Rightarrow\supset \quad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$$

$$\sim\Rightarrow \quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\sim \varphi, \Gamma \Rightarrow \Delta} \qquad \Rightarrow\sim \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim \varphi}$$

if $E$ is $t_1 = t_2$ or $t_2 = t_1$, then

$$\frac{\Gamma, \phi(t_1) \Rightarrow \Delta, \psi(t_1)}{\Gamma, E, \phi(t_2) \Rightarrow \Delta, \psi(t_2)}$$

$$\forall \Rightarrow \quad \frac{\Gamma, \phi(t) \Rightarrow \Delta}{\Gamma, \forall v \phi(v) \Rightarrow \Delta} \quad \Rightarrow \exists \quad \frac{\Gamma \Rightarrow \Delta, \phi(t)}{\Gamma \Rightarrow \Delta, \exists v \phi(v)}$$

For the next two rules the variable $v$ is not allowed occur free in $\Gamma \cup \Delta$.

$$\Rightarrow \forall \quad \frac{\Gamma \Rightarrow \Delta \phi(v)}{\Gamma \Rightarrow \Delta, \forall y \phi(y)} \quad \exists \Rightarrow \quad \frac{\Gamma, \phi(v) \Rightarrow \Delta}{\Gamma, \exists y \phi(y) \Rightarrow \Delta}$$

The final rule is not needed, (it can always be got rid of), but is very useful.

$$\text{Cut} \quad \frac{\Gamma, \phi \Rightarrow \Delta \qquad \Gamma \Delta, \phi}{\Gamma \Rightarrow \Delta}$$

A sentence $\phi$ is derivable from a set of sentences $T$ in the system $G$ iff $T \models \phi$.

The $\Rightarrow$ direction follows from the fact that each rules preserves validity, and the fact that the axioms are valid.

**Definition 9** *A* theory *is a set of sentences with the property that* $T \vdash \phi \Rightarrow \phi \in T$ *(a theory is closed under derivability.*

**Definition 10** *A set* $\Gamma$ *such that* $T = \{\phi | \Gamma \vdash \phi\}$ *is called an axiom set of the theory* $T$*. The elements if* $\Gamma$ *are called axioms.*

**Definition 11** $T$ *is called a* Henkin Theory *if for every sentence* $\exists v.\phi(v)$*, there is a constant* $c$ *such that* $\exists v.\phi(v) \rightarrow \phi(c) \in T$ *(such a* $c$ *is called a* witness *for* $v$ *in* $\phi$*).*

**Definition 12** *$T'$ is an extension of $T$ if $T \subseteq T'$.*

*$T'$ is a conservative extension of $T$ if $T' \cap L = T$ (i.e. all theorems of $T'$ in the language $L$ are already theorems of $T$.*

**Definition 13** *Let $T$ be a theory with language $L$. We add to the language for each sentence $\delta$ of the form $\exists v \phi(v)$, a constant $c_\delta$ such that distinct $\delta$'s yield distinct $c_\delta$s.. The resulting language is $L^*$.*

*$T^*$ is the theory with axiom set $T \cup \{\exists x \phi(x) \supset \phi(c_\delta) | \exists v \phi(v)$ closed , with witness $c_\delta\}$.*

**Lemma 14** *$T^*$ is conservative over $T$.*

Let $\exists x\phi(x) \supset \phi(c)$ be one of the new axioms. Suppose $\Gamma, \exists x\phi(x) \supset \phi(c) \vdash \psi$ where $\psi$ does not contain $c$ and where $\Gamma$ is a set of sentences none of which contain $c$. We show that $\Gamma \vdash \psi$.

$\Gamma \Rightarrow (\exists x\phi(x) \supset \phi(c) \supset \psi$

$\Gamma \Rightarrow (\exists x\phi(x) \supset \phi(y) \supset \psi$ where $y$ is a variable that does not occur in the derivation of the previous line, as we can replace all occurrences of $c$ by $y$, and the proof will still be valid.

$\Gamma, \exists x\phi(x) \supset \phi(y) \Rightarrow \psi$

$\Gamma, \exists x\phi(x) \Rightarrow \phi(y), \psi$

$\Gamma, \exists x\phi(x) \Rightarrow \exists y\phi(y), \psi$ as $y$ does not occur in $\Gamma, \psi$.

$\Gamma, \exists x\phi(x) \supset \exists y\phi(y) \Rightarrow \psi$

But we have $\Rightarrow \exists x\phi(x) \supset \exists y\phi(y)$ so $\Gamma \Rightarrow \exists x\phi(x) \supset \exists y\phi(y)$.

$\Gamma \Rightarrow \psi$ by Cut.

Now let $T^* \vdash \psi$ for a $\psi \in L$. By the definition of derivability there is a finite set of axioms from $T^*$ that are needed to derive $\psi$. These are either sentences from $T$ or of the form $\exists x\phi(x) \supset \phi(c)$. Therefore $T \cup \{\delta_1, \ldots, \delta_n\} \vdash \psi$.

We show $T \vdash \psi$ by induction. For $n = 0$ we are done. For the induction step we use the above.

**Lemma 15** *Define* $T_0 = T; T_{n+1} = (T_n)^*$; *then* $T_\omega = \cup\{T_n | n \geq 0\}$ *is a Henkin theory and* $T_\omega$ *is conservative over* $T$.

Call the language of $T_n$ $L_n$.

$T_\omega$ is a theory: We need to show that $T_\omega \vdash \delta$, if $\psi_0, \ldots \psi_n \vdash \delta$ for certain $\psi_i$ in $T_\omega$.

For each $i \leq n$ $\psi_i \in T_{m_i}$ for some $m_i$.

Let $m = max\{m_i|i \leq n\}$. Since $T_l \subseteq T_{k+1}$ for all $k$, we have $T_{m_i} \subseteq T_m (i \leq n)$

Therefore $T_m \vdash \delta$.

$T_m$ is by definition a theory, so $\delta \in T_m \subseteq T_\omega$.

$T_\omega$ is a Henkin theory. Let $\exists x \phi(x) \in L_\omega$, then $\exists x \phi(x) \in L_n$ for some $n$. By definition $\exists \phi(x) \supset \phi(c) \in T_{n+1}$ for a certain $c$. So $\exists x \phi(x) \supset \phi(x) \in T_\omega$.

$T_\omega$ is conservative over $T$. Observer that $T_\omega \vdash \delta$ if $\mathcal{T}_n \vdash \delta$ for some $n$. By induction of $n$, we can establish that $T_\omega \vdash \delta$ if $\mathcal{T} \vdash \delta$.

**Lemma 16 Zorn's Lemma:** *Every partially ordered set in which every chain (i.e. totally ordered subset) has an upper bound contains at least one maximal element.*

**Lemma 17** *Each consistent theory is contained in a maximally consistent theory.*

Consider the set $A$ of conservative extensions of $T$ ordered by inclusion. Each chain has an upper bound, as the union of the chain is a conservative extension of $T$.

Therefore $A$ has a maximal element $T_m$.

$T_m$ is a maximally conservative extension of $T$. $T_m \subseteq T'$ and $T' \in A$ then $T_m = T$.

**Lemma 18** *If $T_m$ is a maximally consistent extension of $T_\omega$, then $T_m$ is a Henkin theory.*

Being a Henkin Theory is persevered under taking a maximally consistent extension as the language remains fixed.

**Theorem 19** *If Γ is consistent, then Γ has a model.*

Let $T = \{\delta | \Gamma \vdash \delta\}$ be the theory of Γ. Any model of $T$ is a model of Γ.

Let $T_m$ be a maximally consistent extension of $T_\omega$, with language $L_m$.

We construct a model of $T_m$ from $T_m$ itself.

The universe $S$ of the model is the closed terms of $T_m$.

For each function symbol $f$ we define a function $\overline{f}$: $A^k \to A = f(t_1 \dots t_k)$.

For each predicate symbol $P$ we define a relation $\overline{P}$: $A^k$ by $(t_1 \dots t_k) \in \overline{P}$ iff $T_m \vdash P(t_1, \dots, t_k)$.

For each constant symbol $c$ we define $\overline{c} = c$.

This is not a model as equality is not interpreted by real equality.

We create the real model by taking the quotient of this model by $t \sim s$, where $t \sim s$iff $t \sim s \in T_m$.

We need to show that $t \sim s$ is an equivalence relation, i.e. symmetric, transitive and reflexive.

We need to show that all predicates and functions are congruent on $\sim$.

This follows from the axioms for equality.

We call this model $\mathfrak{A}$.

We now show that if $\mathfrak{A} \models \phi$ then $T_m \vdash \phi$ by induction on $\phi$

The base cases are obvious by the definition of the model.

The induction step for propositional connectives follow from the right elimination rules and cut.

The induction step for $\forall$ is the following.

Given $\mathfrak{A} \models \phi(c)$ for all $c \in A$ iff $T_m \vdash \phi(c)$ for all $c$ in $A$, show

n $\mathfrak{A} \models \forall\phi(v)$ iff $T_m \vdash \forall\phi(v)$.

We assume the left, and try to derive the right.

$\mathfrak{A} \models \forall\phi(v)$. Consider the Henkin sentence $\exists x.\neg\phi(x) \supset \neg\phi(c)$.

This is equivalent to $\phi(c) \supset \forall x.\phi(x)$. We have that $\mathfrak{A} \models \phi(c)$ iff $T_m \vdash \phi(c)$, so by one application of $\supset\Rightarrow$ and cut we can derive that $T_m \vdash \forall x.\phi(x)$ as required.

The opposite direction follows from $\forall \Rightarrow$ and cut by instantiating with the $c$ in question.

**Theorem 20**

$$\Gamma \vdash \phi \textit{ iff } \Gamma \models \phi$$

The left to right part is Soundness. We now consider

$$\Gamma \models \phi \text{ implies } \Gamma \vdash \phi$$

We consider the contrapositive:

$$\Gamma \nvdash \phi \text{ implies } \Gamma \not\models \phi$$

$\Gamma \nvdash \phi$ is equivalent to $\Gamma, \neg\phi$ is consistent.

We use the earlier theorem to create a structure $\mathfrak{M}$ where $\mathfrak{M} \models \Gamma, \neg\phi$.

By construction , $\mathfrak{M} \models \Gamma$ and $\mathfrak{M} \models \neg\phi$. Therefore $\mathfrak{M} \not\models \phi$. Therefore we have a model of $\Gamma$ which is not a model of $\phi$ as required.