Completeness of a Gentzen System for First Order Logic

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A first order language \( L \) is a set of function symbols, predicate symbols and constant symbols. Each function and predicate symbol has a positive number associated with it, its arity
\[
\#(f) = n.
\]

Given a language \( L \) we have a notion of a structure for \( L \).

A *structure* for \( L \) is a pair \( \mathcal{M} = \langle M, F \rangle \) where \( M \) is a non-empty set and \( F \) is a operation on the domain \( L \) such that

- if \( P \in L \) is a \( n \)-ary predicate symbol the
  \[
  R^\mathcal{M} \subseteq M^n;
  \]

- if \( f \in L \) is a \( n \)-ary function symbol the
  \[
  f^\mathcal{M} : M^n \to M;
  \]

- if \( c \in L \) is a constant symbol the
  \[
  c^\mathcal{M} \in M;\]
One often write $\mathcal{M}$ as $\langle M, R^\mathcal{M}, \ldots, f^\mathcal{M}, \ldots, c^\mathcal{M} \rangle$.

**Definition 1** The terms of $L$ form the smallest set of expressions containing the variables, $x, y, z, \ldots$, all constant symbols in $L$ and closed under the formation rule: if $t_1, \ldots, t_n$ are terms of $L$ and if $f \in L$ is an $n$-ary function symbol then the expression $f(t_1, \ldots, t_n)$ is a term of $L$. A closed term is a term in which no variable appears.

**Definition 2** An atomic formula of $L$ is an expression of either of then two forms:

$$(t_1 = t_2) \quad P(t_1, \ldots, t_n)$$

where in the first case $t_1$ and $t_2$ are terms of $L$. In the second case $P \in L$ is any $n$-ary predicate symbol and $t_1, \ldots, t_n$ are terms of $L$.

**Definition 3** The first order formulas of $L$ form the smallest set of expressions containing the
atomic formulas and closed under the following formation rules:

- If $\phi, \psi$ are formulas so are the expressions.
  
  $$ \neg \phi, \quad (\phi \lor \psi) \quad (\phi \land \psi) \quad (\phi \supset \psi) $$

- If $\phi$ is a formula and $v$ is a variable, the $(\exists v \phi)$ and $(\forall v \phi)$ are formulas.

**Definition 4** The set of $FV(\phi)$ free variables of a formula $\phi$ is defined as follows:

- If $\phi$ is an atomic formula, the $FV(\phi)$ is just the set of variables appearing in the expression $\phi$.

- $FV(\neg \phi) = FV(\phi)$

- $FV(\phi \lor \psi) = FV(\phi \land \psi) = FV(\phi \supset \psi) = FV(\phi) \cup FV(\psi)$
• \( FV(\exists v \phi) = FV(\forall v \phi) = FV(\phi) - \{v\} \)

**Definition 5** A first order sentence is a formula without any free variables.

**Definition 6** Let \( \mathcal{M} = \langle M, \ldots \rangle \) be a structure for a language \( L \). An assignment in \( \mathcal{M} \) is a function \( s \) with domain the set of variables of \( L \) and range a subset of \( M \). We think of \( s \) as assigning a meaning \( s(v) \) to the variables \( v \). We can then define for each term \( t \) of \( L \), a function \( t^\mathcal{M} \) which maps assignments to elements of \( M \).

Let \( M \) be given. For a term \( t \) of \( L \) we define \( t^\mathcal{M} \) as follows:

• If \( t \) is a constant symbol \( c \) then \( t^\mathcal{M}(s) = c^\mathcal{M} \) for all \( s \).

• if \( t \) is a variable \( v \), then \( t^\mathcal{M}(s) = s(v) \) for all \( s \),
if $t$ is a term $f(t_1, \ldots, t_n)$ then for all $s$ define

$$t^s = f^s(t_1^s(s), \ldots, t_n^s(s)).$$

**Definition 7** In the following definition we use $s(\frac{a}{c})$ for the assignment $s'$ which agrees with $s$ except that $s'(v) = a$.

Let $\mathcal{M}$ be an $L$ structure. We define a relation

$$\mathcal{M} \models \phi[s]$$

(read the assignment $s$ satisfies the formula $\phi$ in $\mathcal{M}$) for all assignments $s$ and all formulas $\phi$ as follows:

- $\mathcal{M} \models (t_1 = t_2)[s]$ iff $t_1^s(s) = t_2^s(s)$,

- $\mathcal{M} \models P(t_1, \ldots, t_n)[s]$ iff $(t_1^s(s), \ldots, t_n^s(s)) \in P^\mathcal{M}$,
• $M \models \neg \phi[s]$ iff not $M \models \phi[s]$,

• $M \models (\phi \lor \psi)[s]$ iff $M \models \phi[s]$ or $M \models \psi[s]$,

• $M \models (\phi \land \psi)[s]$ iff $M \models \phi[s]$ and $M \models \psi[s]$,

• $M \models (\phi \supset \psi)[s]$ iff either not $M \models \phi[s]$ or else $M \models \psi[s]$,

• $M \models (\forall v \phi)[s]$ iff for all $a \in M$, $M \models \phi[s(\frac{a}{c})]$,

• $M \models (\forall v \phi)[s]$ iff there exists an $a \in M$, such that $M \models \phi[s(\frac{a}{c})]$.

**Definition 8** When $\phi$ is a sentence, (i.e. has no free variables), we write $M \models \phi$ for $M \models \phi[s]$ as the truth of this does no depend on $s$. 
A structure $\mathcal{M}$ is a model of a set of sentences $\Phi$ if $\mathcal{M} \models \phi$ for all $\phi \in \Phi$. 
First order logic has as connectives $\land$, $\lor$, $\supset$ and $\sim$. It can be axiomatized in a Gentzen system as follows. Gentzen System Sequents $\Gamma \Rightarrow \Delta$ should be read $\land \Gamma \supset \lor \Delta$, where $\Gamma \cup \Delta$ is a finite set of formulas.

**Axioms**

$$\Gamma, \varphi \Rightarrow \Delta, \varphi$$

$$\Gamma, \varphi \Rightarrow \Delta, t = t (t \text{ any term})$$

**Rules of Inference**

\[
\begin{align*}
\Rightarrow \land & \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \land \psi} \\
\Rightarrow \lor & \quad \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \lor \psi, \Gamma \Rightarrow \Delta} \\
\Rightarrow \supset & \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \supset \psi, \Gamma \Rightarrow \Delta} \\
\Rightarrow \sim & \quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\sim \varphi, \Gamma \Rightarrow \Delta} \\
\Rightarrow & \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim \varphi}
\end{align*}
\]
if $E$ is $t_1 = t_2$ or $t_2 = t_1$, then
\[
\frac{\Gamma, \phi(t_1) \Rightarrow \Delta, \psi(t_1)}{\Gamma, E, \phi(t_2) \Rightarrow \Delta, \psi(t_2)}
\]
\[
\forall \Rightarrow \frac{\Gamma, \phi(t) \Rightarrow \Delta}{\Gamma, \forall v \phi(v) \Rightarrow \Delta} \Rightarrow \exists \frac{\Gamma \Rightarrow \Delta, \phi(t)}{\Gamma \Rightarrow \Delta, \exists v \phi(v)}
\]
For the next two rules the variable $v$ is not allowed occur free in $\Gamma \cup \Delta$.
\[
\Rightarrow \forall \frac{\Gamma \Rightarrow \Delta \phi(v)}{\Gamma \Rightarrow \Delta, \forall y \phi(y)} \exists \Rightarrow \frac{\Gamma, \phi(v) \Rightarrow \Delta}{\Gamma, \exists y \phi(y) \Rightarrow \Delta}
\]
The final rule is not needed, (it can always be got rid of), but is very useful.
\[
\text{Cut} \quad \frac{\Gamma, \phi \Rightarrow \Delta \quad \Gamma \Delta, \phi}{\Gamma \Rightarrow \Delta}
\]
A sentence $\phi$ is derivable from a set of sentences $T$ in the system $G$ iff $T \models \phi$.

The $\Rightarrow$ direction follows from the fact that each rule preserves validity, and the fact that the axioms are valid.
Definition 9 A theory is a set of sentences with the property that $T \vdash \phi \Rightarrow \phi \in T$ (a theory is closed under derivability.

Definition 10 A set $\Gamma$ such that $T = \{ \phi | \Gamma \vdash \phi \}$ is called an axiom set of the theory $T$. The elements if $\Gamma$ are called axioms.

Definition 11 $T$ is called a Henkin Theory if for every sentence $\exists v. \phi(v)$, there is a constant $c$ such that $\exists v. \phi(v) \rightarrow \phi(c) \in T$ (such a $c$ is called a witness for $v$ in $\phi$).
**Definition 12** $T'$ is an extension of $T$ if $T \subseteq T'$.

$T'$ is a conservative extension of $T$ if $T' \cap L = T$ (i.e. all theorems of $T'$ in the language $L$ are already theorems of $T$.
**Definition 13** Let $T$ be a theory with language $L$. We add to the language for each sentence $\delta$ of the form $\exists v \phi(v)$, a constant $c_\delta$ such that distinct $\delta$'s yield distinct $c_\delta$s. The resulting language is $L^*$.

$T^*$ is the theory with axiom set $T \cup \{ \exists x \phi(x) \supset \phi(c_\delta) | \exists v \phi(v) \text{ closed} , \text{ with witness } c_\delta \}$. 


Lemma 14 $T^*$ is conservative over $T$.

Let $\exists x \phi(x) \supset \phi(c)$ be one of the new axioms. Suppose $\Gamma, \exists x \phi(x) \supset \phi(c) \vdash \psi$ where $\psi$ does not contain $c$ and where $\Gamma$ is a set of sentences none of which contain $c$. We show that $\Gamma \vdash \psi$.

$$\Gamma \Rightarrow (\exists x \phi(x) \supset \phi(c) \supset \psi)$$

$$\Gamma \Rightarrow (\exists x \phi(x) \supset \phi(y) \supset \psi)$$ where $y$ is a variable that does not occur in the derivation of the previous line, as we can replace all occurrences of $c$ by $y$, and the proof will still be valid.

$$\Gamma, \exists x \phi(x) \supset \phi(y) \Rightarrow \psi$$

$$\Gamma, \exists x \phi(x) \Rightarrow \phi(y), \psi$$

$$\Gamma, \exists x \phi(x) \Rightarrow \exists y \phi(y), \psi$$ as $y$ does not occur in $\Gamma, \psi$.  

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\[ \Gamma, \exists x \phi(x) \supset \exists y \phi(y) \Rightarrow \psi \]

But we have \( \Rightarrow \exists x \phi(x) \supset \exists y \phi(y) \) so \( \Gamma \Rightarrow \exists x \phi(x) \supset \exists y \phi(y) \).

\[ \Gamma \Rightarrow \psi \text{ by Cut.} \]

Now let \( T^* \vdash \psi \) for a \( \psi \in L \). By the definition of derivability there is a finite set of axioms from \( T^* \) that are needed to derive \( \psi \). These are either sentences from \( T \) or of the form \( \exists x \phi(x) \supset \phi(c) \). Therefore \( T \cup \{ \delta_1, \ldots, \delta_n \} \vdash \psi \).

We show \( T \vdash \psi \) by induction. For \( n = 0 \) we are done. For the induction step we use the above.
Lemma 15 Define $T_0 = T; T_{n+1} = (T_n)^*$; then
$T_\omega = \cup \{T_n | n \geq 0\}$ is a Henkin theory and $T_\omega$ is
conservative over $T$. 
Call the language of $T_n \ L_n$.

$T_\omega$ is a theory: We need to show that $T_\omega \vdash \delta$, if $\psi_0, \ldots, \psi_n \vdash \delta$ for certain $\psi_i$ in $T_\omega$.

For each $i \leq n$ $\psi_i \in T_{m_i}$ for some $m_i$.

Let $m = \max\{m_i | i \leq n\}$. Since $T_i \subseteq T_{k+1}$ for all $k$, we have $T_{m_i} \subseteq T_m(i \leq n)$

Therefore $T_m \vdash \delta$.

$T_m$ is by definition a theory, so $\delta \in T_m \subseteq T_\omega$.

$T_\omega$ is a Henkin theory. Let $\exists x \phi(x) \in L_\omega$, then $\exists x \phi(x) \in L_n$ for some $n$. By definition $\exists x \phi(x) \supset \phi(c) \in T_{n+1}$ for a certain $c$. So $\exists x \phi(x) \supset \phi(x) \in T_\omega$.

$T_\omega$ is conservative over $T$. Observer that $T_\omega \vdash \delta$ if $T_n \vdash \delta$ for some $n$. By induction of $n$, we can establish that $T_\omega \vdash \delta$ if $T \vdash \delta$. 

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Lemma 16 Zorn’s Lemma: Every partially ordered set in which every chain (i.e. totally ordered subset) has an upper bound contains at least one maximal element.
Lemma 17 Each consistent theory is contained in a maximally consistent theory.

Consider the set $A$ of conservative extensions of $T$ ordered by inclusion. Each chain has an upper bound, as the union of the chain is a conservative extension of $T$.

Therefore $A$ has a maximal element $T_m$.

$T_m$ is a maximally conservative extension of $T$. $T_m \subseteq T'$ and $T' \in A$ then $T_m = T$.  

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Lemma 18 If $T_m$ is a maximally consistent extension of $T_\omega$, then $T_m$ is a Henkin theory.

Being a Henkin Theory is persevered under taking a maximally consistent extension as the language remains fixed.
Theorem 19 If $\Gamma$ is consistent, then $\Gamma$ has a model.

Let $T = \{\delta | \Gamma \vdash \delta\}$ be the theory of $\Gamma$. Any model of $T$ is a model of $\Gamma$.

Let $T_m$ be a maximally consistent extension of $T_\omega$, with language $L_m$.

We construct a model of $T_m$ from $T_m$ itself.

The universe $S$ of the model is the closed terms of $T_m$.

For each function symbol $f$ we define a function $\overline{f}: A^k \to A = f(t_1 \ldots t_k)$.

For each predicate symbol $P$ we define a relation $\overline{P}: A^k$ by $(t_1 \ldots t_k) \in \overline{P} \text{ iff } T_m \vdash P(t_1, \ldots, t_k)$.

For each constant symbol $c$ we define $\overline{c} = c$. 

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This is not a model as equality is not interpreted by real equality.

We create the real model by taking the quotient of this model by \( t \sim s \), where \( t \sim s \) iff \( t \sim s \in T_m \).

We need to show that \( t \sim s \) is an equivalence relation, i.e. symmetric, transitive and reflexive.

We need to show that all predicates and functions are congruent on \( \sim \).

This follows from the axioms for equality.

We call this model \( \mathcal{A} \).
We now show that if $\mathcal{A} \models \phi$ then $T_m \vdash \phi$ by induction on $\phi$.

The base cases are obvious by the definition of the model.

The induction step for propositional connectives follow from the right elimination rules and cut.

The induction step for $\forall$ is the following.

Given $\mathcal{A} \models \phi(c)$ for all $c \in A$ iff $T_m \vdash \phi(c)$ for all $c$ in $A$, show

$n \mathcal{A} \models \forall \phi(v)$ iff $T_m \vdash \forall \phi(v)$.
We assume the left, and try to derive the right.

\[ \mathcal{A} \models \forall \phi(v) \]. Consider the Henkin sentence
\[ \exists x. \neg \phi(x) \supset \neg \phi(c) \).

This is equivalent to \( \phi(c) \supset \forall x. \phi(x) \). We have that \( \mathcal{A} \models \phi(c) \) iff \( T_m \vdash \phi(c) \), so by one application of \( \supset \Rightarrow \) and cut we can derive that \( T_m \vdash \forall x. \phi(x) \) as required.

The opposite direction follows from \( \forall \Rightarrow \) and cut by instantiating with the \( c \) in question.
Theorem 20
\[ \Gamma \vdash \phi \text{ iff } \Gamma \models \phi \]

The left to right part is Soundness. We now consider
\[ \Gamma \models \phi \text{ implies } \Gamma \vdash \phi \]

We consider the contrapositive:
\[ \Gamma \not\vdash \phi \text{ implies } \Gamma \not\models \phi \]

\[ \Gamma \not\vdash \phi \text{ is equivalent to } \Gamma, \neg \phi \text{ is consistent.} \]

We use the earlier theorem to create a structure $\mathcal{M}$ where $\mathcal{M} \models \Gamma, \neg \phi$.

By construction, $\mathcal{M} \models \Gamma$ and $\mathcal{M} \models \neg \phi$. Therefore $\mathcal{M} \not\models \phi$. Therefore we have a model of $\Gamma$ which is not a model of $\phi$ as required.